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Admissibility and Event-Rationality

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Admissibility and Event-Rationality*

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Abstract

Brandenburger et al. (2008) establish epistemic foundations for admissibility, or the avoidance of weakly dominated strategies, by using lexicographic type structures and the notion of rationality and common assumption of rationality (RCAR). Their negative result that RCAR is empty whenever the type structure is complete and continuous suggests that iterated admissibility (IA) requires players to have prior knowledge about each other, and therefore is a strong solution concept, not at the same level as iterated elimination of strongly dominated strategies (IEDS). We follow an alternative approach, using standard type structures and the notion of event-rationality. We characterize the set of strategies that are generated under event-rationality and common belief of event-rationality (RCBER) and show that, in a complete structure, it consists of the strategies that are admissible and survive iterated elimination of dominated strategies (Dekel and Fudenberg (1990)). By requiring that agents believe that themselves are E-rational at each level of mutual belief we construct and characterize RCBeER and show that in a complete structure it generates the IA strategies. Contrary to the negative result in Brandenburger et al. (2008), we show that RCBER and RCBeER are nonempty in complete, continuous and compact type structures, therefore providing an epistemic criterion for IA.

Keywords: Epistemic game theory; Admissibility; Iterated weak dominance; Common Knowledge; Rationality; Completeness.

1 Introduction

As noted by Samuelson (1992) and many others, there is an intrinsic impossibility in dealing with common knowledge of admissibility in games, which is known as the inclusion-exclusion problem. The reason is that a strategy is admissible if and only if it is a best response to a conjecture with full support. If we capture knowledge by the support of the agent's belief and assume that

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she is rational, that is, she optimizes given her belief, then playing an admissible strategy implies that she does not know much: she must necessarily consider all strategies of the other players as possible, including the strategies that are not admissible. So she cannot know that her opponents play admissible strategies, that is, cannot exclude from consideration their inadmissible strategies.

The most appealing approach to dealing with this issue is provided by Brandenburger et al. (2008), henceforth BFK. Using lexicographic probability systems (LPS) and the notion of assumption, BFK separate "knowledge" about who plays admissible strategies from the support of the conjecture that is used when choosing a best response. In particular, BFK work with lexicographic type structures, where a type of a player, say Ann, is associated with an LPS $(\mu_0, \dots, \mu_{n-1})$ over the strategy-type pairs of the other player, say Bob. The first conjecture, μ_0, is Ann's primary hypothesis. If she is not able to decide between two of her strategies according to the marginal of μ_0 over Bob's strategies, then she moves on to her secondary hypothesis, μ_1, and so on. That is, Ann looks for a strategy that is a lexicographic best response to the marginals of $(\mu_0, \dots, \mu_{n-1})$ over Bob's strategies. Her LPS has full support if the union of the supports of $\mu_0, \dots, \mu_{n-1}$ is equal to the set of all strategy-type pairs of Bob. A strategy-type of Ann is called rational if the associated LPS has full support and the strategy is a lexicographic best response to the marginals of $(\mu_0, \dots, \mu_{n-1})$ over Bob's strategies.

BFK replace the notion of probability one belief with that of assumption. In particular, Ann *assumes* an event E at level j if E is assigned probability 1 by each of the measures $\mu_1, \dots, \mu_j$ and probability zero by the remaining measures $\mu_{j+1}, \dots, \mu_{n-1}$. Ann assumes an event E if she assumes it at some level j. Hence, rational Ann considers everything to be possible when choosing an admissible strategy (because the union of the supports of her measures is the entire state space) and at the same time "knows" (through assumption) that Bob is rational, even though Bob's rationality is described by an event which is a strict subset of the state space. Common knowledge of admissibility is then approximated by the notion of rationality and common assumption of rationality (RCAR).

However, RCAR can be empty. In fact, BFK shows that RCAR is empty when the type structure is complete and continuous. One reason seems to be that the LPS's always contain finitely many measures, allowing only finitely many statements of the type "Ann assumes that Bob assumes ..." to be assumed. To see this, note that BFK's notion of rationality implies that the supports of the agent's measures form a partition of the state space. Suppose that the set describing $m+1$ rounds of mutual assumption of rationality is a strict subset of the set describing m rounds. If the agent assumes the latter set at level j, he has to assume the former set at a lower level, thus "losing" some measures. Eventually, the agent's assumption reaches the "bottom", or her primary hypothesis. One immediate way out may be to allow for infinitely many measures. BFK mention such an alternative, but do not pursue it. Yang (2009) proposes a weaker notion of assumption, that handles some measurability issues in BFK, and obtains a possibility result. Keisler (2009) and Lee (2009) independently show that RCAR is not empty in a complete structure if continuity is dropped.

All the aforementioned papers use lexicographic type structures. We propose an alternative approach that employs standard type structures, so that each type is associated with a unique measure. Moreover, instead of assumption we use standard probability 1 belief. As a result, we avoid problems such as the violation of the monotonicity property (by assumption and weak assumption) or continuity and technicalities arising from the use of lexicographic structures. Our only departure from the standard model is that we use event-rationality instead of rationality and this is enough to provide epistemic foundations for both the solution concept proposed by Dekel and Fudenberg (1990) ($S^\infty W$) and iterated admissibility (IA). Finally, by modifying only the notion of rationality and not that of belief it is easier to compare the epistemic conditions of $S^\infty W$ and IA with those of iterated elimination of dominated strategies (IEDS), as they share the same type space. This is not necessarily the case with LPS-based approaches.

In order to provide some intuition about event-rationality, note that if a strategy s^a of Ann's is rational then it is a best response to some conjecture, $v \in \Delta(S^b)$, where S^b is the set of Bob's strategies. If s^a is inadmissible and therefore weakly dominated by some strategy σ^a, then both give the same payoff for all strategies of Bob on the support of v while σ^a is strictly better than s^a for all conjectures with support on the complement of the support of v. Hence, whenever Ann chooses an admissible strategy, it is as if she optimizes given her conjecture, as usual, but when she is totally indifferent between two strategies she compares them using a measure with support on the difference between S^b and the support of her conjecture. In other words, she "breaks ties" using the event S^b. Whenever these conditions are met, s^a is called S^b-rational.

But there is nothing particular about S^b when defining event-rationality. Formally, let E^b be a set of Bob's strategies. We say that s^a is E^b-rational if s^a is a best response to some conjecture of Ann, and Ann uses E^b to break ties. In particular, if another strategy σ^a gives the same payoff as s^a for any of Bob's strategies that are considered possible given Ann's conjecture, then s^a must be better under some conjecture with support on the difference between E^b and the support of Ann's initial conjecture. In other words, Ann is confident in trusting her belief, just like any other rational agent. But if two of her strategies are equivalent under her belief, she chooses the one that is also optimal for strategies outside her belief but inside the tie-breaking set.

Given $E = E^a \times E^b$, we can define rationality and common belief of event-rationality (RCBER) in the standard way, as the intersection of infinitely many events: Ann is E^b-rational and Bob is E^a-rational. Ann is certain (assigns probability one to the event) that Bob is E^a-rational and Bob is certain that Ann is E^b-rational; Ann is certain that Bob is certain that Ann is E^b-rational. And so on. Common belief of extended event-rationality (RCBeER) is constructed if on the procedure above we also require that at each level of mutual belief, Ann believes that she is E-rational, (breaks ties using E), where E is the set of strategies played by the types of Bob she considers possible. And similarly for Bob.

Since the tie-breaking sets need not be the supports of conjectures, we do not run into any inclusion-exclusion problems in the construction. Moreover, we have a degree of flexibility that resembles lexicographic beliefs in that the tie-breaking sets need not be fixed as we move to higher order levels of mutual knowledge.

Our results are as follows. Using $E = S^a \times S^b$ as the set of all strategies, we characterize the strategies that are compatible with RCBER by a solution concept, hypo-admissible sets (HAS), which is related to the self-admissible sets (SAS) of BFK but it is neither weaker or stronger. In a complete structure, RCBER produces the set of strategies that survive one round of elimination of non admissible strategies followed by iterated elimination of strongly dominated strategies ($S^\infty W$). We characterize RCBeER with a solution concept we call hypo-iteratively admissible sets (HIA). In a complete type structure, the resulting set of strategies is precisely the set of iterated admissible strategies (IA). We then show that strategies played under RCBeER constitute an SAS, but the converse is not necessarily true, meaning that the RCBeER construction is more restrictive than the RCAR construction of BFK. Nevertheless, we show that the RCBER and the RCBeER are nonempty whenever the type structure is complete, continuous and compact, whereas the RCAR is empty in a complete and continuous (lexicographic) type structure when the agent is not indifferent.

Our approach provides an alternative, effective and simple perspective in dealing with common "knowledge" of admissibility in games. The solution to the inclusion-exclusion problem lies in separating what a player knows from the strategies that she includes in her conjectures. This separation can also be obtained with LPS-based approaches as in BFK, Brandenburger (1992), Stahl (1995) and Yang (2009). But LPS-based approaches may add technical elements that are not necessarily relevant for the issue.[1] For instance, BFK's impossibility result suggests that IA is a solution concept that requires that the players are experienced enough with each other so that the type structure used to describe their beliefs is not complete (Brandenburger and Friedenberg (forthcoming)). In other words, it suggests that IA is to be viewed as a strong solution concept, that is not at the same level as iterated elimination of dominated strategies (IEDS) but rather closer to Nash equilibria, whose epistemic conditions require incomplete type structures (see Aumann and Brandenburger (1995) and Barelli (2009)). But this suggestion is an artifact of the technical details of BFK's LPS-based approach. In fact, RCBeER is more restrictive than RCAR, and it is nonempty in a complete, continuous and compact type structure.

[1] A simple example is that lexicographic type structures typically fail to be compact, whereas the universal type structure, without lexicographic beliefs, is compact.

1.1 Related Literature

Bernheim (1984) and Pearce (1984) provide epistemic foundations for the iteratively undominated strategies via the concept of rationality and common belief in rationality. Admissibility, or the avoidance of weakly dominated strategies, has a long history in decision and game theory (see Kohlberg and Mertens (1986)). However, Samuelson (1992) shows that common knowledge of admissibility is not equivalent to iterated admissibility and does not always exist. Foundations for the $S^{\infty}W$ strategies (Dekel and Fudenberg (1990)) are provided by Börgers (1994) (using approximate common knowledge), Brandenburger (1992) (using lexicographic probability systems (Blume et al. (1991)) and 0-level belief) and Ben-Porath (1997) (in extensive form games). Stahl (1995) defines the notion of lexicographic rationalizability and shows that it is equivalent to iterated admissibility.

BFK use lexicographic probability systems and characterize rationality and common assumption of rationality (RCAR) by the solution concept of self-admissible sets. They show that rationality and m-th order assumption of rationality is characterized by the strategies that survive $m+1$ rounds of elimination of inadmissible strategies. Finally, RCAR is empty in a complete and continuous lexicographic type structure when the agent is not indifferent. Hence, although the IA set can be captured by RmAR, for big enough m (note that games are finite), BFK do not provide an epistemic criterion for IA. Yang (2009) provides an epistemic criterion for IA, with an analogous version of BFK's RCAR, that makes use of a weaker notion of "assumption". Keisler (2009) and Lee (2009) independently show that the emptiness of RCAR can be overcome if one drops continuity. The message from Yang (2009), Keisler (2009) and Lee (2009) is that continuity strengthens the notion of caution implied by fully supported LPS. The notion of caution implied by Event-Rationality is independent of continuity.

The paper is organized as follows. In the following section we illustrate the differences between the various notions of rationality and belief through examples. In Sections 3 and 4 we set up the framework and provide the relevant definitions, including event-rationality, RCBER and RCBeER. In Section 5 we characterize RCBER and show that RmBER (m rounds of mutual belief) generates $S^{\infty}W$, for big enough m. In Section 6 we characterize RCBeER, show that it is more restrictive than RCAR of BFK and show that RmBeER generates the IA set, for big enough m. In Section 7 we show that RCBER and RCBeER are always nonempty in compact, complete and continuous type structures, therefore providing epistemic criteria for $S^{\infty}W$ and IA. Finally, some decision theoretic remarks are presented in the Appendix.

2 Examples

In order to illustrate the differences between the BFK approach and that of the present paper, consider the following game from Samuelson (1992) and BFK. There are two players, Ann and Bob.

		1	[1]
		L	R
1	U	$1,1$	$0,1$
[1]	D	$0,2$	$1,0$

From Bernheim (1984) and Pearce (1984) we know that rationality and common belief of rationality (RCBR) is characterized by the best response sets (BRS) and, in a complete structure, the strategies that survive iterated deletion of strongly dominated strategies.[2] Can we get a similar result for the admissible strategies and the iteratively admissible strategies if we modify the notions of belief and of rationality? Recall that a strategy is admissible if and only if it is a best response to a full support measure (no action of the other player is excluded). Then, the obvious solution is to specify that rationality incorporates full support beliefs.

But such a specification does not always work. In the game above, if Ann is rational, she assigns positive probability to Bob playing L and R. If Bob is rational, he assigns positive probability to Ann playing U and D. Hence, Bob plays L. If Ann knows that Bob is rational, she assigns positive probability only on Bob playing L. But then, Ann is not rational! In other words, the modified RCBR set is empty for this game.

BFK solve the problem by introducing lexicographic probabilities.[3] Suppose Ann's primary hypothesis assigns probability 1 to Bob playing L, and her secondary hypothesis assigns probability 1 to Bob playing R. Bob's primary hypothesis assigns 1 on U and his secondary hypothesis assigns 1 on D. Then, Bob playing L is rational because he is indifferent between L and R given his primary measure, but strictly prefers L given his secondary measure.[4] Ann playing U is rational because U is the best response given her primary measure. She assumes that Bob is rational, because she considers Bob playing L infinitely more likely than Bob playing R. Similarly, Bob assumes that Ann is rational. As a result, rationality and common assumption of rationality (RCAR) is nonempty.

A similar result can be obtained if we use the definition of event-rationality in the context of standard type structures. Suppose Ann's belief assigns probability 1 to Bob playing L and Bob's

[2] $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

[3] The same example is analyzed at great length in BFK.

[4] In the terminology of BFK, the associated sequence of payoffs under L is lexicographically greater than the sequence under R.

belief μ assigns probability 1 to Ann playing U. Bob playing L is S^a-rational because he is playing best response given his beliefs and whenever he is indifferent between L and R, L is better given a conjecture with support $S^b \setminus \operatorname{supp} \mu$. Similarly, Ann is S^b-rational. Finally, Ann believes that Bob is S^a-rational and Bob believes that Ann is S^b-rational. Hence, rationality and common belief of event-rationality (RCBER) is nonempty.

In the game above RCAR and RCBER produce the same strategies because the IA and the $S^\infty W$ sets are equal. However, this is not always true. Consider the following game which illustrates the difference between RCBER (which yields the $S^\infty W$ set) and RCBeER (which yields the IA set).

	L	R
U	$1,0$	$1,3$
M	$0,2$	$2,2$
D	$0,4$	$1,1$

Since D is strongly dominated, no S^b-rational type plays that strategy. In a complete structure though, Ann's S^b-rational types play U or M and Bob's S^a-rational types play L or R. For example, Ann's type is S^b-rational if she plays U, while assigning probability 1 to Bob playing L. Ann's type is also S^b-rational if she plays M, while assigning probability 1 to Bob playing R. Moreover, for both U and M there are S^b-rational types of Ann's who assign positive probability to S^a-rational types of Bob playing L or R. And similarly for Bob. In other words, these types of Ann believe the event "Bob is S^a-rational", Bob's types believe the event "Ann is S^b-rational", and so on for any finite order of beliefs about beliefs. Hence, $S^a \times S^b$-rationality and common belief of $S^a \times S^b$-rationality (RCBER) yields the $S^\infty W$ set, $\{\mathrm{U}, \mathrm{M}\} \times \{\mathrm{L}, \mathrm{R}\}$.

Suppose we repeat the same procedure but also require that at every level m of mutual belief, Ann believes that she is E-rational, where E is the set of strategies played by Bob's types who survive m levels of mutual belief (and similarly for Bob).Then, imposing common belief of rationality will give us RCBeER. Which strategies are generated by RCBeER? The first round of RCBeER yields the set of S^b-rational types for Ann and S^a-rational types for Bob, just like RCBER.

But the second round of RCBeER requires that, in addition to believing the event "Bob is rational", each of Ann's S^b-rational types uses the strategies played by Bob's rational types as her tie-breaking set. Then all types playing L are excluded. To see this, note that any S^a-rational type playing L must assign probability 1 to Ann playing M. Although Bob's type believes the event "Ann is rational" and Ann's rational types play either U or M, Bob playing L is not $\{\mathrm{U}, \mathrm{M}\}$-rational. For example, given Bob's beliefs (probability 1 on M) he is indifferent between L and R. Yet, if he compares L and R against U (which is the tie-breaking set $\{\mathrm{U}, \mathrm{M}\}$ minus the support of his beliefs, M), L is worse. Once all types playing L are excluded, only types who believe S^a-rational types playing R survive in the third round of RCBeER . Hence, playing U is no longer S^a-rational for Ann. Finally, RCBeER yields the IA set, which is $\{M\} \times \{R\}$.

3 Set Up

Let (S^a, S^b, π^a, π^b) be a two player finite strategic form game, with $\pi^a : S^a \times S^b \to \mathbb{R}$, and similarly for b (as usual, a stands for Ann, and b stands for Bob). For any given topological space X, let $\Delta(X)$ denote the space of probability measures defined on the Borel subsets of X, endowed with the weak* topology. We extend π^a to $\Delta(S^a) \times \Delta(S^b)$ in the usual way: $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b)\in S^a\times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$. Similarly for π^b.

3.1 Admissibility and Event-Rationality

The following definition and Lemma are taken from BFK.

Definition 1. *Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is* **weakly dominated** *with respect to $X \times Y$ if there exists $\sigma^a \in \Delta(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$ and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is* **admissible** *with respect to $X \times Y$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is admissible.*

Lemma 1. *A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there exists $\sigma^b \in \Delta(S^b)$, with supp $\sigma^b = Y$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

Lexicographic beliefs have been used in dealing with the inclusion-exclusion issue identified by Samuelson (1992) (see BFK, Brandenburger (1992), Stahl (1995) and Yang (2009)). Instead, we will use the following construction. For a given conjecture $v \in \Delta(S^b)$, let $\sigma^a \sim_{\text{supp } v} s^a$ denote that the mixed strategy $\sigma^a \in \Delta(S^a)$ satisfies $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for every $s^b \in \text{supp } v$.

Definition 2. *Let $E^b \subset S^b$. A strategy $s^a \in S^a$ is E^b-rational if there exists a conjecture $v \in \Delta(S^b)$ such that s^a is a best response to v and, if $E^b \setminus \text{supp } v \neq \varnothing$, then for each mixed strategy $\sigma^a \in \Delta(S^a)$ such that $\sigma^a \sim_{\text{supp } v} s^a$ there exists a conjecture $v' \in \Delta(S^b)$ with supp $v' = E^b \setminus \text{supp } v$ such that $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. Likewise for b.*

The idea is that Ann uses the set E^b to break ties: whenever she has a conjecture $v \in \Delta(S^b)$ over Bob's choices under which s^a is an optimal choice and s^a is outcome equivalent to a (mixed) strategy σ^a in supp v, Ann uses E^b as the "tie-breaking hypothesis": there has to exist a conjecture v' in what is in E^b and not considered by v that supports the choice of s^a. Ann is fully confident in her assessment v and in her best response s^a to v as long as there is no σ^a that is outcome equivalent to s^a in supp v. In that case, her probabilistic assessments are irrelevant, for whichever other conjecture $\hat{v}$ with supp $\hat{v}$ = supp v would not help Ann breaking ties between s^a and σ^a. In that case, Ann uses the tie breaking set E^b.

It is important to note that, although the "tie-breaking conjecture" supported in $E^b \setminus \text{supp } v$ is a secondary measure that Ann uses to guide her choices, it does not play the role of a secondary hypothesis in a lexicographic framework. If s^a is indifferent to σ^a according to v, but not outcome equivalent in supp v, then there is no need to break ties. The following lemma shows the connection between admissibility and event-rationality.

Lemma 2. *A strategy $s^a \in S^a$ is admissible with respect to $S^a \times E^b$ if and only if it is E^b-rational for v such that supp $v \subseteq E^b$.*

Proof. Suppose that s^a is E^b-rational for v such that supp $v \subseteq E^b$. Suppose there exists $\sigma^a \in \Delta(S^a)$ with $\pi(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in E^b$, with strict inequality for some $s^b \in E^b$. Then, $s^a \sim_{\text{supp } v} \sigma^a$, which implies that there exists v' with supp $v' = E^b \setminus \text{supp } v$, such that $\pi(s^a, v') \geq \pi(\sigma^a, v')$, a contradiction. Conversely, if s^a is admissible with respect to $S^a \times E^b$ then it is a best response to a conjecture v with supp $v = E^b$, so $E^b \setminus \text{supp } v = \varnothing$, and s^a is E^b-rational. □

In particular, a strategy s^a is admissible if and only if it is S^b-rational, for the requirement supp $v \subseteq S^b$ is trivially met. This will be the case in our analysis in later sections, where we take E to be equal to $S^a \times S^b$.

3.2 Type Structures and Beliefs

Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

Definition 3. *An (S^a, S^b)-based type structure is a structure*

$$\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle,$$

where $\lambda^a : T^a \to \Delta(S^b \times T^b)$, and similarly for b. Members of T^a, T^b are called types, members of $S^a \times T^a \times S^b \times T^b$ are called states.

A type structure is **complete** when λ^a and λ^b are surjective and it is **continuous** when these mappings are continuous.[5] Because the strategy spaces are finite, it is without loss of generality to work with complete and continuous type structures, where in addition T^a and T^b are compact spaces (Mertens and Zamir (1985)). Such type structures are called complete, continuous and compact type structures.

[5]Our definition of completeness is more restrictive than that of BFK; as with compactness and continuity below, it follows directly from working with universal type structures, as in Mertens and Zamir (1985).

As defined, a type t^a of Ann has beliefs only about Bob's types and strategies. However, we will also need to describe Ann's beliefs about her own strategies. So let $\nu^a : S^a \times T^a \to \Delta(S^a \times T^a \times S^b \times T^b)$ be given by

$$\nu^a(s^a, t^a) = \delta_{(s^a,t^a)} \otimes \lambda^a(t^a)$$

where $\delta_{(s^a,t^a)}$ is the Dirac measure concentrated at (s^a, t^a), and define ν^b similarly. Hence, a strategy-type pair (s^a, t^a) has beliefs $\lambda(t^a)$ over Bob's types and strategies and assigns probability 1 on choosing strategy s^a (and having type t^a). In other words, each strategy-type pair of Ann knows the strategy she takes.

Fix an event $E \subseteq S^a \times T^a \times S^b \times T^b$ and write

$$B^a(E) = \{(s^a, t^a) \in S^a \times T^a : \nu^a(s^a, t^a)(E) = 1\}$$

as the set of strategy-type pairs that are certain of the event E. Note that this is the standard definition of certainty (as 1-belief), which (unlike assumption and weak assumption, defined below) satisfies monotonicity: if Ann is certain of E and $E \subset F$ then Ann is also certain of F. The belief operator B^a is a slight generalization of the standard operator $B_0^a(E) = \{t^a \in T^a : \lambda^a(t^a)(E) = 1\}$, where $E \subseteq S^b \times T^b$.

The LPS-based approach, on the other hand, uses the following construction. Let $\mathcal{L}^+(X)$ be the space of fully supported LPS's over X, that is, the space of finite sequences $\sigma = (\mu_0, \ldots, \mu_{n-1})$, for some integer n, where $\mu_i \in \Delta(X)$ and $\bigcup_{i=0}^{n-1} \text{supp } \mu_i = X$. In addition, the measures μ_i in σ are required to be non-overlapping, that is, mutually singular. A lexicographic type structure is a type structure where $\lambda^a : T^a \to \mathcal{L}^+(S^b \times T^b)$, and similarly for b. An event E is **assumed** if and only if the closure of the event is equal to the union of the supports of j levels of the player's LPS. That is, there is a level j such that the player assigns probability one to the event E for all of his/her hypothesis up to level j, and assigns probability zero to the event for all of his/her hypothesis of levels higher than j. Yang (2009) uses a weaker notion that allows the levels higher than j to assign positive (and strictly smaller than 1) weights to the event. The use of lexicographic beliefs is to be contrasted with our use of standard beliefs.

3.3 RCBER - Common Belief of Event-Rationality

Definition 4. *Let $E^b \subset S^b$. Say that a strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is E^b-rational if s^a is E^b-rational under the conjecture $marg_{S^b}\lambda^a(t^a)$.*

Let R_1^a be the set of E^b-rational strategy-type pairs (s^a, t^a). For finite m, define R_m^a inductively by

$$R_{m+1}^a = R_m^a \cap B^a(S^a \times T^a \times R_m^b).$$

Similarly for b, using $E^a \subset S^a$.

Note that $B^a(S^a \times T^a \times R^b_m)$ are the strategy-type pairs of Ann that are certain of R^b_m.

Definition 5. *If $(s^a, t^a, s^b, t^b) \in R^a_{m+1} \times R^b_{m+1}$, say there is event-rationality and mth-order belief of event-rationality (RmBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} R^a_m \times \bigcap_{m=1}^{\infty} R^b_m$ say there is event-rationality and common belief of event-rationality (RCBER) at this state.*

In words, there is RCBER at a state if Ann is E^b-rational, Ann believes that Bob is E^a-rational, Ann believes that Bob believes that Ann is E^b-rational, and so on. Similarly for Bob. Believing that Bob is E^a-rational means that Ann is certain that Bob only chooses strategies that are best responses to Bob's conjectures that Ann considers possible, and that Bob breaks ties using E^a.

Note that for a strategy-type pair (s^a, t^a) to belong to R^a_m the following conditions are satisfied. Strategy s^a is a best response to $v = \text{marg}_{S^b}\lambda^a(t^a)$, and $\lambda^a(t^a)(R^b_{m-1}) = 1$, and whenever $\sigma^a \sim_{\text{supp } v} s^a$ there exists a conjecture v' in $E^b \backslash \text{supp } v$ (if $E^b \setminus \text{supp } v \neq \varnothing$) for which $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. Notice that Ann is certain that the conjectures of Bob are of the form $v = \text{marg}_{S^a}\lambda^b(t^b)$, for $t^b \in \text{proj}_{T^b} R^b_{m-1}$, and knows that, for each such conjecture, Bob breaks each tie using some v' in $E^b \setminus \text{supp } v$. We show below that this flexibility implies that the set of strategies compatible with RCBER are the ones that survive one round of elimination of inadmissible strategies followed by iterated elimination of strongly dominated strategies.

3.4 RCBeER - Common Belief of extended Event-Rationality

Fix a type structure, set $E^b \subseteq S^b$ and let $R^a(E^b)$ be the set of strategy-type pairs of Ann who are E^b-rational. For example, if R^a_1 is the set of S^b-rational strategy-type pairs (s^a, t^a) we have that $R^a_1 = R^a(S^b)$. For a given $E = E^a \times E^b$, let $\overline{R}^a_1 = R^a(E^b)$ and define, for finite m,

$$\overline{R}^a_{m+1} = \overline{R}^a_m \cap B^a(R^a(\text{proj}_{S^b}\overline{R}^b_m) \times \overline{R}^b_m).$$

Similarly for b, using $E^a \subset S^a$.

Note that $B^a(R^a(\text{proj}_{S^b}\overline{R}^b_m) \times \overline{R}^b_m)$ are the strategy-type pairs of Ann that are certain that Bob's strategy-type pairs are in $\overline{R}^b_m$ and that Ann herself is $\text{proj}_{S^b}\overline{R}^b_m$-rational.

Consider $\overline{R}^a_3$ for instance. This is Ann's set of strategy-type pairs that are E^b-rational, believe that they are $\text{proj}_{S^b}(\overline{R}^b_1)$-rational *and* $\text{proj}_{S^b}(\overline{R}^b_2)$-rational, together with the corresponding two levels of mutual belief. That is, the strategies in $\text{proj}_{S^a}\overline{R}^a_3$ must satisfy three tie-breaking conditions. It follows that the strategies in $\text{proj}_{S^a}\overline{R}^a_m$ must satisfy m tie-breaking conditions.

A strategy-type pair of Ann that belongs to $\overline{R}^a_{m+1}$ believes that Bob is m orders rational (as in RCBER) *and* that she, herself, is E-rational, where E is the set of strategies played by the strategy-type pairs of Bob that are m orders rational.

Definition 6. *If $(s^a, t^a, s^b, t^b) \in \overline{R}^a_{m+1} \times \overline{R}^b_{m+1}$, say there is extended event-rationality and mth-order belief of extended event-rationality (RmBeER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \bigcap_{m=1}^{\infty} \overline{R}^b_m$ say there is extended event-rationality and common belief of extended event-rationality (RCBeER) at this state.*

Obviously, RCBeER requires more in terms of beliefs than RCBER, which implies that if they are both defined for the same tie-breaking set $E = E^a \times E^b$, we have RCBeER $\subseteq$ RCBER.

4 Solution Concepts

4.1 Self-Admissible and Hypo-Admissible Sets

If Ann's tie-breaking set is S^b and Bob's tie-breaking set is S^a then all event-rational types play admissible strategies. If, in addition, there is common belief of event-rationality, then the solution concept is that of a hypo-admissible set (HAS) that we define below. We compare the HAS with several solution concepts that have been proposed in the literature. But first a definition.

Definition 7. *Say that r^a supports s^a given Q^b if there exists some $\sigma^a \in \Delta(S^a)$ with $r^a \in supp\ \sigma^a$ and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in Q^b$. Write $su_{Q^b}(s^a)$ for the set of $r^a \in S^a$ that supports s^a given Q^b. Likewise for b.*

This is a generalization of the definition in BFK of the support of a strategy s^a, which they denote $\text{su}(s^a)$. In particular, $\text{su}_{S^b}(s^a) = \text{su}(s^a)$.

BFK characterize rationality and common assumption of rationality (RCAR) by the solution concept of a self-admissible set (SAS).

Definition 8. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an SAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{S^b}(s^a)$, then $r^a \in Q^a$.*

Likewise for b.

In particular, BFK show that the projection of the RCAR into $S^a \times S^b$ is an SAS. Conversely, given an SAS $Q^a \times Q^b$, there is a type structure such that the projection of RCAR into $S^a \times S^b$ is equal to $Q^a \times Q^b$. BFK discuss the need for the third requirement in the definition of an SAS. In particular, consider the weak best response sets (WBRS), which does not include a restriction on convex combinations.

Definition 9. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is a WBRS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is not strongly dominated with respect to $S^a \times Q^b$.*

Likewise for b.

An "almost" characterization of the WBRS is obtained if, as in Brandenburger (1992) and Börgers (1994), common assumption of rationality is relaxed to common belief at level 0 of rationality (RCB0R) (that is, believing E means $\mu_0(E) = 1$, where μ_0 is the first measure of the agent's LPS). More specifically, on the one hand the projection of RCB0R into $S^a \times S^b$ is a WBRS. On the other hand, given a WBRS $Q^a \times Q^b$, there is a type structure such that $Q^a \times Q^b$ is contained in (but not necessarily equal to) the projection of RCB0R into $S^a \times S^b$.[6]

We are now ready to introduce the solution concept of hypo-admissible sets (HAS).

Definition 10. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an HAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$.*

  For each $s^a \in Q^a$ there is nonempty $Q_0 \subseteq Q^b$ such that

- *s^a is admissible with respect to $S^a \times Q_0$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{Q_0}(s^a)$ and r^a is admissible with respect to $S^a \times S^b$ then $r^a \in Q^a$.*

Likewise for b.

Note that the first two properties for a WBRS are equivalent to the first two properties for an HAS and they are implied by the first two properties for an SAS. Hence, the SAS and the HAS are always WBRS but the opposite does not hold. Moreover, an SAS is not necessarily an HAS and an HAS is not necessarily an SAS. The differences between the HAS and the SAS can be further illustrated by the following two solution concepts. The first is $S^\infty W$, the set of strategies that survive one round of deletion of inadmissible strategies followed by iterated deletion of strongly dominated strategies (Dekel and Fudenberg (1990)).

Definition 11. *Set $SW_1^i = S_1^i$, for $i = a, b$ be the set admissible strategies and define inductively for $m \geq 1$,*

$$SW_{m+1}^i = \{s^i \in SW_m^i : s^i \textit{ is not strongly dominated with respect to } SW_m^a \times SW_m^b\}.$$

Let $S^\infty W = \bigcap_{m=1}^\infty SW_m^a \times \bigcap_{m=1}^\infty SW_m^a$.

The second is the set of strategies that survive iterated deletion of weakly dominated strategies, the IA set.

[6] See Section 11 in BFK.

Definition 12. *Set $S_0^i = S^i$ for $i = a, b$ and define inductively*

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is admissible with respect to } S_m^a \times S_m^b\}.$$

A strategy $s^i \in S_m^i$ is called m-admissible. A strategy $s^i \in \bigcap_{m=0}^{\infty} S_m^i$ is called iteratively admissible (IA).

We then have that the $S^\infty W$ set is both an HAS and a WBRS (but not an SAS) and the IA set is an SAS and a WBRS (but not a HAS). The following game from Section 2 illustrates the various definitions.

	L	R
U	1, 0	1, 3
M	0, 2	2, 2
D	0, 4	1, 1

The IA set is $\{M\} \times \{R\}$. It is an SAS but not an HAS, because although $L \in \text{su}_{\{M\}}(R)$ and L is admissible, it does not belong to the IA set. Moreover, $S^\infty W = \{U, M\} \times \{L, R\}$ is an HAS but not an SAS, because L is not admissible with respect to $\{U, M\}$. That is, in a sense the SAS captures IA whereas the HAS captures $S^\infty W$.

4.2 Generalized Self-Admissible and Hypo-Iteratively Admissible Sets

In Section 5 we show that HAS characterizes RCBER with $E = S$. With a view to obtain a characterization of RCBeER and to relate it to the concepts presented above, we introduce the following two solution concepts.

Definition 13. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an $SAS_{P^a \times P^b}$ if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{P^b}(s^a)$ and r^a is admissible with respect to $S^a \times S^b$, then $r^a \in Q^a$.*

Likewise for b.

This is a generalization of the SAS, since the only difference is that the support $\text{su}_{P^b}(s^a)$ is with respect to an abstract set P^b, not S^b. This means that the SAS is equivalent to the $\text{SAS}_{S^a \times S^b}$.[7]

[7]Note that if $r^a \in \text{su}_{S^b}(s^a)$ and s^a is admissible, then r^a is also admissible. Hence, the third condition for a $\text{SAS}_{S^a \times S^b}$ is identical to the third condition for a SAS.

Moreover, if $Q^a \times Q^b \subseteq P^a \times P^b$ then an $\text{SAS}_{Q^a \times Q^b}$ is also an $\text{SAS}_{P^a \times P^b}$, but the reverse may not hold. This means that for any $P^a \times P^b$, an $\text{SAS}_{P^a \times P^b}$ is also an SAS. Moreover, an $\text{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is also an HAS.

Definition 14. *A set $Q^a \times Q^b$ is a hypo-iteratively admissible (HIA) set if there exist sequences of sets $\{W_i^a\}_{i=0}^{\infty}$, $\{W_i^b\}_{i=0}^{\infty}$, with $W_0^a = S^a$, $W_0^b = S^b$, such that for each $m \geq 0$,*

- *each $s^a \in W_{m+1}^a$ is admissible with respect to $S^a \times W_m^b$ and belongs to W_m^a,*
- *for any k, m, where $k \geq m$, if $s^a \in W_{k+1}^a$, $r^a \in su_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$, then $r^a \in W_{m+1}^a$,*
- *there is k such that for all $m \geq k$, $W_m^a = Q^a$.*

Likewise for b.

The HIA sets resemble the IA set, with the only difference that one starts with a subset of admissible strategies and always includes the strategies that are equivalent (in the sense of su_Q) to strategies that survive subsequent rounds. Moreover, the HIA can be thought of as an analogue of the best response set (BRS).[8] If we replace admissible with strongly undominated in the definition of HIA then we get a BRS. Conversely, each BRS $Q^a \times Q^b$ can be written as a modified HIA (just set $W_i^a = Q^a$ and $W_i^b = Q^b$ for all $i \geq 1$).

5 Characterization of RCBER

In the definitions of RCBER and RCBeER, the initial tie-breaking set is left unspecified. In what follows, we keep the convention that $E = S$. Our first result shows that HAS characterizes RCBER.

Proposition 1. *Let R_1^a be the set of strategy-type pairs (s^a, t^a) who are S^b-rational and R_1^b the set of strategy-type pairs (s^b, t^b) who are S^a-rational.*

(i) Fix a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is an HAS.

(ii) Fix an HAS $Q^a \times Q^b$. Then there is a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} R_m^b$.

Proof. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is empty, then the conditions for HAS are satisfied, so suppose that it is nonempty and fix $s^a \in Q^a = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a$. Then, for some t^a, (s^a, t^a) is S^b-rational, so s^a is optimal under $v = \text{marg}_{S^b} \lambda^a(t^a)$. Let $Q_0 = \text{supp } v$.

[8] Recall that $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

We have that s^a is admissible with respect to $Q_0 = \text{supp } v$, which is a subset of $Q^b = \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. From Lemma 2 $(s^a, t^a) \in R_1^a$ implies that s^a is admissible.

Suppose $s^a \in Q^a$, $r^a \in \text{su}_{\text{supp } v}(s^a)$ and r^a is admissible with respect to $S^a \times S^b$. From Lemma D.2 in BFK, r^a is optimal under v whenever $(s^a, t^a) \in R_1^a$.[9] Suppose $\sigma^a \sim_{\text{supp } v} r^a$. If it is the case that σ^a dominates r^a on $S^b \setminus \text{supp } v$ then r^a is not admissible with respect to $S^a \times S^b$, a contradiction. Hence, $(r^a, t^a) \in R_1^a$. The same is true for all R_m^a, hence the third property for an HAS is satisfied.

For part (ii) fix an HAS $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to $Q_{s^a} \subseteq Q^b$, there is a v with $\text{supp } v = Q_{s^a}$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q_{s^a}}(s^a)$ (Lemma D.4 in BFK).[10] Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that $\text{supp } \lambda^a(s^a) = \{(s^b, s^b) : s^b \in Q_{s^a}\}$, $\text{supp } \lambda^b(s^b) = \{(s^a, s^a) : s^a \in Q_{s^b}\}$, and $v = \text{marg}_{S^b}\lambda^a(s^a)$ for the v found above (and similarly for $\text{marg}_{S^a}\lambda^b(s^b)$).

First, we show that for each $s^a \in Q^a$, (s^a, s^a) is S^b-rational. By construction, s^a is optimal under $v = \text{marg}_{S^b}\lambda^a(s^a)$. Suppose that for some σ^a we have that $s^a \sim_{\text{supp } v} \sigma^a$. If it is the case that s^a is dominated by σ^a under $S^b \setminus \text{supp } v$, then s^a is not admissible with respect to S^b, a contradiction. Hence, (s^a, s^a) is S^b-rational and $Q^a \subseteq \text{proj}_{S^a} R_1^a$. Suppose $(r^a, t^a) \in R_1^a$, where $t^a = s^a$. Then, $r^a \in \text{su}_{Q_{s^a}}(s^a)$ and r^a is admissible with respect to Q_{s^a}. From Lemma 2, r^a is admissible. From the definition of an HAS this implies that $r^a \in Q^a$ and $Q^a = \text{proj}_{S^a} R_1^a$. Applying similar arguments we have that $Q^b = \text{proj}_{S^b} R_1^b$.

By construction, each $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b) which consists of S^a-rational types, hence t^a believes R_1^b and (s^a, s^a) believes $S^a \times T^a \times R_1^b$. This implies that $R_2^a = R_1^a$ and likewise for b. Thus, $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m, by induction. Since $\text{proj}_{S^a} R_1^a \times \text{proj}_{S^b} R_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. ☐

That is, the strategies consistent with RCBER are the hypo-admissible strategies according to the definition of an HAS. In a complete structure, m rounds of mutual belief generate the $SW_m^a \times SW_m^b$ strategies.

Proposition 2. *Fix a complete structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m,*

$$proj_{S^a} R_m^a \times proj_{S^b} R_m^b = SW_m^a \times SW_m^b.$$

Proof. From Lemma 2 we have that $(s^a, t^a) \in R_1^a$ implies s^a is admissible. Conversely, since we have a complete structure, if s^a is admissible then there exists t^a such that $(s^a, t^a) \in R_1^a$. Hence,

[9] Lemma D.2 specifies that if F is a face of a polytope P and $x \in F$, then $su(x) \subseteq F$, where $su(x)$ is the set of points that support x. The geometry of polytopes is presented in Appendix D in BFK.

[10] Lemma D.4 specifies that if x belongs to a strictly positive face of a polytope P, then $su(x)$ is a strictly positive face of P.

$\text{proj}_{S^a} R_1^a = S_1^a = SW_1^a$ and $\text{proj}_{S^b} R_1^b = S_1^b = SW_1^b$. Suppose that for up to m we have that $\text{proj}_{S^a} R_m^a = SW_m^a$ and $\text{proj}_{S^b} R_m^b = SW_m^b$. Suppose $s^a \in SW_{m+1}^a$. Then, $s^a \in SW_m^a = \text{proj}_{S^a} R_m^a$. Because s^a is not strongly dominated with respect to $SW_m^a \times SW_m^b$, it is also not strongly dominated with respect to $S^a \times SW_m^b$. Hence, there is a v with supp $v \subseteq SW_m^b$ under which s^a is optimal. Hence, we can take (s^a, t^a) with supp $\lambda^a(t^a) \subseteq R_m^b$ and $\text{marg}_{S^b} \lambda^a(t^a) = v$. Because s^a is admissible with respect to S^b, (s^a, t^a) is S^b-rational. Since $(s^a, t^a) \in B^a(S^a \times T^a \times R_m^b)$ and $R_m^b \subseteq R_k^b$, $1 \leq k \leq m$, we have that $(s^a, t^a) \in R_{m+1}^a$ and $s^a \in \text{proj}_{S^a} R_{m+1}^a$.

Suppose $s^a \in \text{proj}_{S^a} R_{m+1}^a$. Then, $s^a \in SW_m^a = \text{proj}_{S^a} R_m^a$ and supp $\text{marg}_{S^b} \lambda^a(t^a) \subseteq SW_m^b = \text{proj}_{S^b} R_m^b$. Because s^a is optimal under v, where supp $v \subseteq SW_m^b$, s^a is not strongly dominated with respect to SW_m^b and therefore $s^a \in SW_{m+1}^a$. □

6 Characterization of RCBeER

The following two Propositions show that RCBeER is characterized by the HIA and RmBeER generates the IA set in a complete type structure, for big enough m.

Proposition 3.

(i) Fix a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is an HIA set.

(ii) Fix an HIA set $Q^a \times Q^b$. Then there is a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proof. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is empty, then the conditions for an HIA set are satisfied, so suppose that it is nonempty.

Set $W_m^a = \text{proj}_{S^a} \overline{R}_m^a$ for $m \geq 1$ and likewise for b. From Lemma 2, all strategies in $\text{proj}_{S^b} \overline{R}_{m+1}^a$ are admissible with respect to $S^a \times W_m^b$ and, by construction, belong to $\text{proj}_{S^b} \overline{R}_m^a$.

Suppose that for some k, m, where $k \geq m$, we have that $s^a \in W_{k+1}^a = \text{proj}_{S^b} \overline{R}_{k+1}^a$, $r^a \in su_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$. This implies that for some t^a, $(s^a, t^a) \in \overline{R}_{k+1}^a$ and supp $\text{marg}_{S^b} \lambda^a(t^a) \subseteq W_k^b$. Because $r^a \in su_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$ we have that $(r^a, t^a) \in B^a(R^a(\text{proj}_{S^b} \overline{R}_l^b) \times \overline{R}_l^b)$ for all $l \leq m$. Note that (r^a, t^a) believing $R^a(\text{proj}_{S^b} \overline{R}_l^b) \times S^b \times T^b$ is equivalent to $(r^a, t^a) \in R^a(\text{proj}_{S^b} \overline{R}_l^b)$. To see that the last part is true, note that $r^a \in W_m^b$ implies that r^a is admissible with respect to each $\text{proj}_{S^b} \overline{R}_l^b$. But this implies that $(r^a, t^a) \in R^a(\text{proj}_{S^b} \overline{R}_l^b)$. Hence, $(r^a, t^a) \in \overline{R}_m^a$ and therefore $(r^a, t^a) \in \overline{R}_{m+1}^a$. The third condition is satisfied because $\text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is nonempty and the strategies are finite.

Fix an HIA set $Q^a \times Q^b$ and construct the following type structure. For each $m \geq 1$, for each $s^a \in W_m^a$, find the measure $v(s^a, m)$ with support on W_{m-1}^b such that r^a is a best response to

$v(s^a,m)$ if and only if $r^a \in \text{su}_{W^b_{m-1}}(s^a)$. This is possible because of Lemma D.4 in BFK. The type $t^a(s^a,m)$ has a marginal $v(s^a,m)$ on S^b and assigns positive probability only to strategy-types $(s^b, t^b(s^b, m-1))$, for $s^b \in W^b_{m-1}$. Finally, assign to each $s^a \in S^a$ type $t^a(r^a, 0)$ which is equal to $t^a(r^a, k)$, for some $r^a \in W^a_k$, $k > 0$.

We now show that RCBeER generates the HIA set. For $m = 1$, we show that $\text{proj}_{S^a}\overline{R}^a_1 = W^a_1$. Suppose that $s^a \in W^a_1$. Because s^a is admissible and a best response to $v(s^a, 1)$, we have $(s^a, t^a(s^a, 1)) \in \overline{R}^a_1$ and $s^a \in \text{proj}_{S^a}\overline{R}^a_1$. Suppose $r^a \in \text{proj}_{S^a}\overline{R}^a_1$. Then, r^a is a best response to some measure $v(s^a, k+1)$, $k \geq 0$, for $s^a \in W^a_{k+1}$ and $r^a \in \text{su}_{W^b_k}(s^a) \cap W^a_0$. Moreover, $(r^a, t^a(s^a, k+1))$ is S^b-rational and therefore admissible, hence by the second property for an HIA set we have that $r^a \in W^a_1$.

Assume that for up to m, $\text{proj}_{S^a}\overline{R}^a_m = W^a_m$ and $\text{proj}_{S^b}\overline{R}^b_m = W^b_m$. Suppose that $s^a \in W^a_{m+1}$. By construction, s^a is a best response to $v(s^a, m+1)$, which has a support of $W^b_m = \text{proj}_{S^b}\overline{R}^b_m$, and it is admissible with respect to $S^a \times W^b_m$. This implies that $(s^a, t^a(s^a, m+1)) \in B^a(R^a(\text{proj}_{S^b}\overline{R}^b_k) \times \overline{R}^b_k)$ for all $k \leq m$. Hence, $s^a \in \text{proj}_{S^a}\overline{R}^a_{m+1}$. Suppose $r^a \in \text{proj}_{S^a}\overline{R}^a_{m+1}$. Then, $r^a \in \text{proj}_{S^a}\overline{R}^a_m = W^a_m$. By construction, the only measures that have support which is a subset of W^b_m are measures that are associated with strategies s^a that belong to W^a_{k+1}, where $k+1 > m$. Hence, r^a is a best response to some measure $v(s^a, k+1)$, $k+1 > m$, for $s^a \in W^a_{k+1}$ and $r^a \in \text{su}_{W^b_k}(s^a)$. Moreover, $(r^a, t^a(s^a, k+1))$ is $\overline{R}^b_m$-rational and therefore admissible with respect to $S^a \times W^b_m$. Hence, by the second property for an HIA set we have that $r^a \in W^a_{m+1}$.

□

Proposition 4. *Fix a complete type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m,*

$$\textit{proj}_{S^a}\overline{R}^a_m \times \textit{proj}_{S^b}\overline{R}^b_m = S^a_m \times S^b_m.$$

Proof. For $m = 1$, Lemma 2 and a complete structure imply $\text{proj}_{S^a}\overline{R}^a_1 = S^a_1$. Suppose that for up to m we have that $\text{proj}_{S^a}\overline{R}^a_m = S^a_m$ and $\text{proj}_{S^b}\overline{R}^b_m = S^b_m$. Suppose $s^a \in S^a_{m+1}$. Then, $s^a \in S^a_m = \text{proj}_{S^a}\overline{R}^a_m$. Because s^a is admissible with respect to $S^a_m \times S^b_m$, it is also admissible with respect to $S^a \times S^b_m$ and we can take (s^a, t^a) such that supp $\lambda^a(t^a) = \overline{R}^b_m$ and s^a is optimal under $\text{marg}_{S^b}\lambda^a(t^a)$. Note that $(s^a, t^a) \in B^a(R^a(\text{proj}_{S^b}\overline{R}^b_k) \times \overline{R}^b_k)$ for each $k \leq m$, because s^a is admissible with respect to every S^b_k and $\overline{R}^b_m \subseteq \overline{R}^b_k$. Hence, $s^a \in \text{proj}_{S^a}\overline{R}^a_{m+1}$.

Suppose $s^a \in \text{proj}_{S^a}\overline{R}^a_{m+1}$. Then, $s^a \in S^a_m = \text{proj}_{S^a}\overline{R}^a_m$ and supp $\text{marg}_{S^b}\lambda^a(t^a) \subseteq S^b_m = \text{proj}_{S^b}\overline{R}^b_m$. Because $(s^a, t^a) \in R^a(\text{proj}_{S^b}\overline{R}^b_m)$ we have that s^a is admissible with respect to $S^a_m \times S^b_m$ and $s^a \in S^a_{m+1}$.

□

RCAR in BFK is characterized by the SAS and RmAR (m levels of mutual assumption) produces the IA set in a complete structure, for big enough m. Since RmBeER generates the IA set as well, it is important to know what is the relationship between RCAR and RCBeER in terms of the solution concepts they generate. The following Proposition and examples show that RCBeER generates a strict subclass of SAS, hence it is a more restrictive notion than RCAR. However, as we show in the following section, RCBeER and RCBER are always nonempty in a complete, continuous and compact structure, unlike RCAR. Let A^a and A^b be the set of Ann's and Bob's admissible strategies, respectively.

Proposition 5.

(i) Fix a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. *Then* $proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ *is an* $SAS_{A^a\times A^b}$.

(ii) Fix an $SAS_{Q^a\times Q^b}$ $Q^a \times Q^b$. *Then there is a type structure* $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b\rangle$ *with* $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proof. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is empty, then the conditions for $\text{SAS}_{A^a\times A^b}$ are satisfied, so suppose that it is nonempty. Because $\overline{R}_1^b = R_1^b$, Lemma 2 establishes that $s^a \in Q^a$ is admissible. Moreover, $s^a \in Q^a$ is $\text{proj}_{S^b}\overline{R}_m^b$-rational for each m, so it is Q^b-rational for some t^a. From Lemma 2 we then have that s^a is admissible with respect to $S^a \times Q^b$.

Suppose $s^a \in Q^a$, $r^a \in \text{su}_{A^b}(s^a)$ and r^a is admissible. This implies that for any t^a, $(s^a, t^a) \in \overline{R}_1^a \cap \overline{R}_2^a$ implies that supp $\text{marg}_{S^b}\lambda^a(t^a) \subseteq A^b$ and r^a is optimal under $v = \text{marg}_{S^b}\lambda^a(t^a)$ (Lemma D.2 in BFK). Because r^a is admissible we have that $(r^a, t^a) \in \overline{R}_1^a$. For each $m \geq 2$, $(s^a, t^a) \in \overline{R}_m^a$ implies that (s^a, t^a) is $\text{proj}_{S^b}\overline{R}_{m-1}^b$-rational. But then, s^a is optimal under a measure with support $\text{proj}_{S^b}\overline{R}_{m-1}^b$, which is a subset of A^b. Since $r^a \in \text{su}_{A^b}(s^a)$ we have that r^a is optimal under a measure with support $\text{proj}_{S^b}\overline{R}_{m-1}^b$. Hence, (r^a, t^a) is $\text{proj}_{S^b}\overline{R}_{m-1}^b$-rational and $(r^a, t^a) \in \overline{R}_m^a$. Therefore, $r^a \in Q^a$.

For part (ii) fix an $\text{SAS}_{Q^a\times Q^b}$ $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to Q^b, there is a v with supp $v = Q^b$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q^b}(s^a)$ (Lemma D.4 in BFK). Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that supp $\lambda^a(s^a) = \{(s^b, s^b) : s^b \in Q^b\}$ and supp $\lambda^b(s^b) = \{(s^a, s^a) : s^a \in Q^a\}$.

Note that an $\text{SAS}_{Q^a\times Q^b}$ $Q^a \times Q^b$ is an HAS, where $Q_{s^a} = Q^b$. Because the construction of types is the same as in the proof of Proposition 1 and by definition $R_1^a = \overline{R}_1^a$, we can apply the same arguments in order to get $Q^a = \text{proj}_{S^a}\overline{R}_1^a$ and $Q^b = \text{proj}_{S^b}\overline{R}_1^b$.

By construction, each type $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b) which consists of S^a-rational types, hence t^a believes $\overline{R}_1^b$. Moreover, $t^a = s^a$ is admissible with respect to $Q^b = \text{proj}_{S^b}\overline{R}_1^b$, hence (s^a, s^a) is $\overline{R}_1^b$-rational. This implies that $\overline{R}_2^a = \overline{R}_1^a$ and likewise for b. Thus, $\overline{R}^a_m = \overline{R}^a_1$ and $\overline{R}^b_m = \overline{R}^b_1$ for all m, by induction. Since $\text{proj}_{S^a}\overline{R}^a_1 \times \text{proj}_{S^b}\overline{R}^b_1 = Q^a \times Q^b$ we also have $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$. □

In words, for a given type structure, the strategies compatible with RCBeER form a subclass of all of the SAS, and there is a class of SAS (the $Q^a \times Q^b$ sets that are $\text{SAS}_{Q^a \times Q^b}$) whose strategies are compatible with RCBeER for some type structure. Because an $\text{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is an $\text{SAS}_{A^a \times A^b}$ but the converse is not true, Proposition 5 does not provide a characterization of RCBeER. It does show, however, that RCAR, which is characterized by SAS (BFK, Proposition 8.1), is less restrictive than RCBeER.

In fact, the following game provides an example of an SAS that is not an $\text{SAS}_{A^a \times A^b}$ and cannot be generated by RCBeER for any type structure. Hence, RCBeER generates a strict subclass of SAS.

	L	C	R
U	1, 1	2, 1	1, 1
M	2, 2	0, 1	1, 0
D	0, 1	4, 2	0, 0

Note that all strategies except for R are admissible and that $\{\text{U}\} \times \{\text{L}, \text{C}\}$ is an SAS but not an $\text{SAS}_{A^a \times A^b}$. The reason is that D and M are in the support of a mixed strategy (assigning weight 1/2 to each) that is equivalent to U given that Bob plays his admissible strategies L and C, but not given the set of all strategies S^b. Since D and M are not included in $\{\text{U}\} \times \{\text{L}, \text{C}\}$, this is not an $\text{SAS}_{A^a \times A^b}$.

We now argue that $\{\text{U}\} \times \{\text{L}, \text{C}\}$ cannot be the outcome of RCBeER. First, note that if this were the case, the types of Ann included in RCBeER should assign zero probability to Bob playing R. Note also that U is a best response only when $Pr(\text{L}) = \frac{2}{3}$ and $Pr(\text{C}) = \frac{1}{3}$ and, for these conjectures, also M and D are best responses. Hence, any rational type playing U is also rational when playing M and D. Is it possible that M and D are excluded because types playing these strategies are not $\{\text{L}, \text{C}\}$-rational or S^b-rational? No, because M and D are admissible with respect to both $\{\text{L}, \text{C}\}$ and S^b. Hence, under RCBeER, for any type structure, whenever U is included, M and D are included as well.

In the following game all strategies are admissible, hence an SAS is equivalent to an $\text{SAS}_{A^a \times A^b}$.

	L	C	R
U	1, 1	2, 1	1, 1
M	2, 2	0, 1	1, 5
D	0, 1	4, 2	0, 0

The same arguments show that RCBeER cannot produce $\{\text{U}\} \times \{\text{L}, \text{C}\}$ which is both an SAS and an $\text{SAS}_{A^a \times A^b}$ but not an $\text{SAS}_{Q^a \times Q^b}$. Hence, we cannot have a tighter characterization in terms of Proposition 5.

7 Possibility Results for RCBER and RCBeER

Since the games are assumed to be finite, Propositions 2 and 4 suggest that RmBER and RmBeER generate the $S^\infty W$ and IA sets, respectively, for big enough m and when the tie-breaking set E is equal to S. However, an epistemic criterion for $S^\infty W$ and IA has to be the same across all games and therefore independent of m. Below we show that RCBER and RCBeER are nonempty, whatever the tie-breaking sets, whenever the type structure is complete, continuous and compact (and recall that the universal type structure (Mertens and Zamir (1985)) satisfies these properties), hence providing an epistemic criterion for $S^\infty W$ and IA.

Proposition 6. *Fix a complete, continuous and compact type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then RCBER is nonempty for any tie-breaking sets $E^a \times E^b \subset S^a \times S^b$. Likewise, RCBeER is nonempty for any sequence of tie-breaking sets.*

Proof. We first show that R_1^a and R_1^b are nonempty. Take (s^a, t^a), $v = \text{marg}_{S^b} \lambda^a(t^a)$, such that $\text{supp } v = E^b$ and s^a is optimal under v. Because $E^b \setminus \text{supp } v$ is empty, (s^a, t^a) is E^b-rational.

Suppose R_m^a and R_m^b are nonempty. There are two cases. First, $\text{proj}_{S^b} R_m^b \not\subseteq E^b$. Take (s^a, t^a), $v = \text{marg}_{S^b} \lambda^a(t^a)$, such that $\text{supp } v \subseteq \text{proj}_{S^b} R_m^b \setminus E^b$ and s^a is optimal under v. Because $E^b \setminus \text{supp } v$ is empty, (s^a, t^a) is E^b-rational and believes $S^a \times T^a \times R_k^b$ for each $k \leq m$. Hence, $(s^a, t^a) \in R_{m+1}^a$. Second, suppose that $\text{proj}_{S^b} R_m^b \subseteq E^b$. Note that there exists s^a that is admissible with respect to both $S^a \times \text{proj}_{S^b} R_m^b$ and $S^a \times E^b$. This implies that s^a is optimal under some v, where $\text{supp } v = \text{proj}_{S^b} R_m^b$. Take (s^a, t^a), $v = \text{marg}_{S^b} \lambda^a(t^a)$. Then (s^a, t^a) is E^b-rational and believes $S^a \times T^a \times R_k^b$ for each $k \leq m$. Hence, $(s^a, t^a) \in R_{m+1}^a$.

Because T^a is compact, R_1^a is also compact: for any sequence (s_n^a, t_n^a) in R_1^a, we have $s_n^a \in BR(v_n^a)$, where $v_n^a = marg_{S^b} \lambda^a(t_n^a)$ and BR denotes best response. If $(s_n^a, t_n^a) \to (s^a, t^a)$, then $v_n^a \to v^a = marg_{S^b} \lambda^a(t^a)$, implying that $s^a \in BR(v^a)$. Further, since S^b is finite, we can choose a subsequence such that $\text{supp } v_n^a = \text{supp } v_m^a$ and a fortiori $\text{supp } v_n^a = \text{supp } v^a$. Hence, whenever $\sigma^a \sim_{\text{supp } v^a} s^a$, we have $\sigma^a \sim_{\text{supp } v_n^a} s^a$, and hence the existence of a v' in $E^b \setminus \text{supp } v^a$ such that $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. That is, $(s^a, t^a) \in R_1^a$, so it is a closed subset of the compact space $S^a \times T^a$.

Now consider $R_2^a = R_1^a \cap B^a(S^a \times T^a \times R_1^b)$, and pick a convergent sequence (s_n^a, t_n^a) therein, with limit (s^a, t^a). Because R_1^b is closed and λ^a is continuous, we have $\limsup_{t_n^a \to t^a} \lambda^a(t_n^a)(R_1^b) \leq \lambda^a(t^a)(R_1^b)$. Hence $\lambda^a(t^a)(R_1^b) = 1$ because $\lambda^a(t_n^a)(R_1^b) = 1$. Also, E^b-rationality follows from the same argument above, and we conclude that R_2^a is compact. Inductively, R_m^a is compact for all m. It follows that $\bigcap_{m\geq 1} R_m^a \neq \varnothing$ because the family $\{R_m^a\}_{m\geq 1}$ has the finite intersection property: for any finite list $\{m_1, \ldots, m_K\}$ of positive numbers, let $m_{\overline{k}}$ be the largest. Then we know that $R_{m_{\overline{k}}}^a \neq \varnothing$ and it is included in $\bigcap_{k=1}^K R_{m_k}^a$.

For the RCBeER, the same argument as above shows that $\overline{R}_1^a$ and $\overline{R}_1^b$ are nonempty. Assume that $\overline{R}_m^a$ and $\overline{R}_m^b$ are nonempty. Let $\hat{S}_0^i = E^i$ for $i = a, b$ and define inductively $\hat{S}_{m+1}^i = \{s^i \in \hat{S}_m^i :$ s^i is admissible with respect to $\hat{S}_m^a \times \hat{S}_m^b\}$. We then have $\text{proj}_{S^a}\overline{R}_m^a = \hat{S}_m^a$ and $\text{proj}_{S^b}\overline{R}_m^b = \hat{S}_m^b$. Pick $s^a \in \hat{S}_{m+1}^a$. Then, $s^a \in \hat{S}_m^a = \text{proj}_{S^a}\overline{R}_m^a$. Because s^a is admissible with respect to $\hat{S}_m^a \times \hat{S}_m^b$, it is also admissible with respect to $\hat{S}^a \times S_m^b$ and we can take (s^a, t^a) such that supp $\lambda^a(t^a) = \overline{R}_m^b$ and s^a is optimal under $\text{marg}_{S^b}\lambda^a(t^a)$. Note that $(s^a, t^a) \in B^a(R^a(\text{proj}_{S^b}\overline{R}_k^b) \times \overline{R}_k^b)$ for each $k \leq m$, because s^a is admissible with respect to every $\hat{S}_k^b$ and $\overline{R}_m^b \subseteq \overline{R}_k^b$. Then $(s^a, t^a) \in \overline{R}_{m+1}^a$.

Now pick a sequence (s_n^a, t_n^a) in $\overline{R}_m^a$ converging to (s^a, t^a) and repeat the argument above to conclude that $\lambda^a(t^a)(\overline{R}_{m-1}^b) = 1$, and that supp $v_n^a =$ supp v^a, where $v_n^a = marg_{S^b}\lambda^a(t_n^a)$ and $v^a = marg_{S^b}\lambda^a(t^a)$. Whenever $\sigma^a \sim_{\text{supp } v^a} s^a$, there exist conjectures v_j' with $j = 1, \ldots, m$ with supp $v_1' = S^b \setminus \text{supp } v^a$ and supp $v_j' = \text{proj}_{S^b}\overline{R}_{j-1}^b \setminus \text{supp } v^a$ for $j = 2, \ldots, m$ such that $\pi^a(s^a, v_j') \geq \pi^a(\sigma^a, v_j')$, for $j = 1, \ldots, m$. Hence $\overline{R}_m^a$ is compact, and the intersection $\bigcap_{m=1}^\infty \overline{R}_m^a$ is not empty.

The same arguments apply to b.

□

8 Conclusion

We showed that event-rationality can be used to analyze common belief of admissibility in games. In particular, an epistemic criterion for IA can be obtained and it places IA at the same level as IEDS as a solution concept. IA does require that players know more about each other than IEDS does (i.e. event-rationality instead of plain rationality), but it certainly does not require that players know each other's conjectures. The fact that each player can perform the IA procedure on her own by considering that the other players only play admissible strategies, much as each player can perform the IEDS procedure on her own (by considering that the other players only play rational strategies), suggests that the epistemic requirements for IA ought not be much more restrictive than those for IEDS, as we indeed show using RCBeER.

Also, because we adopt a perspective different from LPS-based approaches, our analysis is a straightforward extension of the standard analysis of common knowledge of rationality. That is, by noting that admissibility can be captured by breaking ties outside of one's conjectures, we are able to separate beliefs from conjectures and work with standard type spaces. LPS-based approaches, on the other hand, analyze admissibility from the perspective of fully supported conjectures, and separate beliefs from conjectures by means of lexicographic type spaces.

A Decision Theoretic Properties

In terms of the underlying decision theoretic ideas, let us present a model of preferences that captures the ideas of event-rationality.[11] Let Ω be a finite set of states, C a finite set of consequences and $\mathcal{F} = \{x : \Omega \to \Delta(C)\}$ be the set of acts that a decision maker faces. Let x_ω denote the value of the act x at the state $\omega \in \Omega$, and for each $A \subset \Omega$ let x_A denote the tuple $(x_\omega)_{\omega \in A}$. A preference relation is a binary relation defined on $\mathcal{F}$. Let AA denote the set of preference relations on $\mathcal{F}$ satisfying the axioms in Anscombe and Aumann (1963). For a given preference relation $\succsim$ defined on $\mathcal{F}$, let $\succsim_A$ denote conditional preference given $A \subset \Omega$, that is, $x \succsim_A y$ if there exists $z \in \mathcal{F}$ such that $(x_A, z_{\Omega \backslash A}) \succsim (y_A, z_{\Omega \backslash A})$. Let $N(\succsim)$ be the set of Savage-null events in Ω according to $\succsim$, that is, $x \sim_{N(\succsim)} y$ for every pair of acts $x, y \in \mathcal{F}$. An act is called constant if $x_\omega = c$ for every $\omega \in \Omega$, for some $c \in \Delta(C)$.

Let $\gtrapprox \in AA$ and fix a set $D \subset N(\gtrapprox)$. Put $C(\gtrapprox) = \{\dot{\succsim} \in AA : x \gtrapprox y \Leftrightarrow x \dot{\succsim} y$ for all constant acts x, y and $N(\dot{\succsim})^c = D\}$. Let $\sim$, $\approx$, $\dot{\sim}$ and $\succ$, $>$, $\dot{\succ}$ denote the symmetric and asymmetric parts of $\succsim$, $\gtrapprox$ and $\dot{\succsim}$, respectively. Consider the following property:

Property 1. *A preference relation $\succsim$ on $\mathcal{F}$ satisfies **double checking** of x over y if there exists $\gtrapprox \in AA$ such that $x \succsim y$ if and only if in addition to $x \gtrapprox y$ we also have that if $x_\omega = y_\omega$ for all $\omega \in N(\gtrapprox)^c$ then there exists $\dot{\succsim} \in C(\gtrapprox)$ with $x \dot{\succsim} y$.*

The idea of double checking is that the decision maker uses $\gtrapprox$ to guide her choices of x over y, but whenever $x = y$ on the non Savage-null states according to $\gtrapprox$ she double checks with preference relations $\dot{\succsim}$ whose Savage-null states include the states that are not Savage-null according to $\gtrapprox$. The resulting preference relation is denoted by $\succsim$.

A subjective expected utility is a pair (u, p) where $u : \Delta(C) \to \mathbb{R}$ and $p \in \Delta(\Omega)$ such that the expected utility of act x is given by $\mathbb{E}_p u(x) = \sum_{\omega \in \Omega} u(x_\omega) p(\omega)$. Given a subjective expected utility (u, p) and a set $E \subset \Omega$, we say that x is **event-rational relative to** y if $\mathbb{E}_p u(x) \geq \mathbb{E}_p u(y)$ and if $x_\omega = y_\omega$ for all $\omega \in \text{supp } p$ then there exists a $v \in \Delta(\Omega)$ with $\text{supp } v = E \backslash \text{supp } p$ (provided that $E \backslash \text{supp } p \neq \varnothing$) such that $\mathbb{E}_v u(x) \geq \mathbb{E}_v u(y)$.

Lemma 3. *A preference relation $\succsim$ on $\mathcal{F}$ satisfies double checking of x over y if and only if there exists a subjective expected utility (u, p) and a set $E \subset \Omega$ such that $x \succsim y$ if and only if x is event-rational relative to y.*

Proof. If $\succsim$ satisfies double checking of x over y then there is $\gtrapprox \in AA$ satisfying the conditions in Property 1. Let $\gtrapprox$ be represented by (u, p), and let $E = \text{supp } p \cup D$. Then $x \succsim y$ means that

[11] Because we work with standard type structures, belief is the standard notion captured by events that are not Savage-null - see definitions below. What we establish now is the properties that a preference relation ought to satisfy for it to give rise to event-rational behavior. BFK (Yang (2009)) present axiomatizations of their notion of assumption (weak assumption), whereas the axiomatization of lexicographic beliefs can be found in Blume et al. (1991).

$\mathbb{E}_p u(x) \geq \mathbb{E}_p u(y)$ and if $x = y$ on $N(\gtrapprox)^c$ then there is $\dot{\succsim} \in C(\gtrapprox)$ with $x \mathrel{\dot{\succsim}} y$. Because $\dot{\succsim}$ agrees with $\gtrapprox$ on constant acts, its subjective expected utility representation is of the form (u, v). And we have supp $v = D = E \backslash$supp p and $\mathbb{E}_v u(x) \geq \mathbb{E}_v u(y)$, so $x \succsim y$ if and only if x is event-rational relative to y.

Conversely, if $\succsim$ is such that there exists a subjective expected utility (u, p) and a set $E \subset \Omega$ such that $x \succsim y$ if and only if x is event-rational relative to y then let $\gtrapprox \in AA$ be represented by (u, p), and let $D = E \backslash$supp p. Then $x \succsim y$ means that $\mathbb{E}_p u(x) \geq \mathbb{E}_p u(y)$ and that if $x_\omega = y_\omega$ for all $\omega \in$ supp p, then there exist $v \in \Delta(\Omega)$ with supp $v = D$ with $\mathbb{E}_v u(x) \geq \mathbb{E}_v u(y)$. Let $\dot{\succsim}$ be represented by (u, v), so that it is in $C(\gtrapprox)$ and we are done. □

We say that an act x is event-rational if it is event-rational relative to y for every act y. In the context of normal form games, a strategy s^a can be viewed as an act with state space S^b, and it is straightforward to verify that a strategy s^a is E^b-rational according to Definition 2 if and only if it is E^b-rational as an act.

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