Well-posed formulation of scalar-tensor effective field theory

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Aron Kovacs and HSR arXiv: 2003.04327 (PRL), 2003.08398 (PRD)
HSR arXiv: 2101.11623 (PRD)

Motivation

Detection of gravitational waves from BH (or NS) mergers is an opportunity to perform precision tests of GR in the *strong field* regime.

To do this we need theoretical predictions for how a deviation from GR would affect the gravitational waves emitted in a merger. Focus on BH/BH mergers so looking for deviations from GR in vacuum.

Two problems:

1. Could try to predict using a theory of modified gravity but which theory should we use?
2. To make predictions we need to perform numerical simulations. This requires that the theory admits a *well-posed initial value problem*, i.e., given suitable initial data there should exist a unique (up to diffeos) solution of the equations of motion that depends continuously on the initial data.

Effective field theory

Provides a way of studying (small) deviations from GR that is agnostic about whatever "UV physics" causes this deviation.

Starting from Einstein-Hilbert action, add all possible higher derivative terms consistent with diffeomorphism invariance

$$S = \int d^d x \sqrt{-g} \left(-2\Lambda + R + \alpha R^2 + \beta R_{ab} R^{ab} + \gamma L_{GB} + \ldots \right)$$

where $\alpha, \beta, \gamma \propto L^2$ for some length scale L and $L_{GB} \propto R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2$

Expansion in increasing numbers of derivatives: makes sense provided deviation from GR is *small*.

To be observable, need $L \sim \mathrm{km}$. Seems very unlikely from perspective of fundamental theory! Instead view this just as a framework for parameterising strong field tests of GR, analogous to the PPN formalism.

Field redefinitions

In EFT one can perform field redefinitions

$$g_{ab} \to g_{ab} + aR_{ab} + bRg_{ab} + \dots$$

These can be used to eliminate higher derivative terms in the action that vanish when the equations of motion are satisfied.

This gives

$$S = \int d^d x \sqrt{-g} \left(-2\Lambda + R + \gamma L_{GB} + \dots \right)$$

$d = 4$: L_{GB} topological so need to go to next order in derivatives.

$d > 4$: L_{GB} gives leading EFT corrections to GR and has second order eqs of motion - it is a Lovelock theory. Well-posed?

6 derivatives

Including 6 derivative terms, after field redefinitions one obtains (schematic) Endlich et al 2017

$$S = \int d^4x \sqrt{-g} \left(-2\Lambda + R + \alpha R^3 + \dots\right)$$

New problem: (truncated) equations of motion are higher order in derivatives. Very unlikely to admit a well-posed initial value problem so can't study BH mergers numerically...

(Can study equations perturbatively in α but this can suffer from problems with secular effects gradually accumulating, leading to breakdown of perturbation theory when EFT should remain valid.)

Scalar-tensor theory

The situation improves if we add a scalar field. After field redefinitions, assuming a parity symmetry, action can be written

$$S = \int \frac{d^4x\sqrt{-g}}{16\pi G}\left(-V(\phi) + R + X + \alpha(\phi)X^2 + \beta(\phi)L_{\mathrm{GB}} + \ldots\right)$$

where $X = -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

Attractive features of this theory:

- Leading EFT corrections now start at 4 derivatives.
- L_{GB} can source scalar field: guaranteed deviation from GR for vacuum BHs
- If we neglect terms with more than 4 derivatives then equations of motion are second order - can hope for well posedness!

This theory belongs to Horndeski family of theories (scalar-tensor theories with second order eqs of motion)

Weak coupling

$$S = \int \frac{d^4x\sqrt{-g}}{16\pi G}\left(-V(\phi) + R + X + \alpha(\phi)X^2 + \beta(\phi)L_{\rm GB} + \ldots\right)$$

where $X = -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

Expect $\alpha, \beta \sim L^2$ for some length L ($\rm km$ scale if observable).

Neglect of terms with 6 or more derivatives is justified only if curvature and scalar field derivatives small in units of L^{-1}. This implies 4-derivative terms are small compared to 2-derivative terms. We call this the *weakly coupled* regime. Compatible with strong-field BH dynamics provided BH large compared to L.

Only expect well-posedness at weak coupling e.g. in cosmological solutions, equations for tensor modes can change character from hyperbolic to elliptic at strong coupling. Papallo & HSR 2017 Similarly in spherical collapse solutions. Ripley & Pretorius 2019

Well-posedness

The usual Einstein equation does not admit a well-posed initial value problem because of gauge freedom.

Establishing well-posedness requires finding a "good gauge" and a good way of performing gauge-fixing. (Essential for numerics!)

Simplest choice for vacuum GR is harmonic gauge but this does not work for our theory, even at weak coupling Papallo & HSR 2017, Papallo 2017

Our main result:
By introducing a new family of "modified harmonic" gauges for GR, and a new way of gauge-fixing, we obtain a new formulation of the equations of motion of these theories (and general Horndeski or Lovelock theories) that admits a well-posed initial value problem at weak coupling. Kovacs & HSR 2020

This formulation has been used successfully in numerical simulations of BH mergers East & Ripley 2020

Strong hyperbolicity

A sufficient condition for a well-posed initial value problem is that the eq is *strongly hyperbolic*.

1st order linear constant coefficients system $\partial_t u = M^i \partial_i u + Nu$

$$u(t,x) \propto \int d\xi e^{i\xi_j x^j} e^{(iM^i\xi_i + N)t} \tilde{u}(0,\xi)$$

For convergence of integral demand $||e^{iM^i\xi_i t}|| \leq f(t)$ as $\xi \to \infty$. This implies that $M^i\xi_i$ must be *diagonalizable* with *real eigenvalues* (which fix phase velocities of modes). This is the definition of strong hyperbolicity (even when coefficients are not constant). (*Weakly* hyperbolic: real evals but not diagonalisable.)

Second order systems: reduce to first order and apply this definition. Nonlinear eqs: apply definition to linearisation around general background (weakly coupled in our case).

Harmonic gauge in GR

The Einstein equation is not hyperbolic because of the diffeomorphism gauge symmetry. To make it hyperbolic we must fix the gauge.

Define

$$H^\mu = g^{\nu\rho}\nabla_\nu\nabla_\rho x^\mu = g^{\nu\rho}\Gamma^\mu_{\nu\rho}$$

The equation $H^\mu = 0$ defines *harmonic gauge*. This gauge condition can always be imposed. Now define

$$E_{\mu\nu} \equiv R_{\mu\nu} - \partial_{(\mu}H_{\nu)} = -\frac{1}{2}g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu} + \ldots$$

where ellipsis depends on g and ∂g but not $\partial^2 g$. The *harmonic gauge Einstein equation* is $E_{\mu\nu} = 0$. This is strongly hyperbolic.

Harmonic gauge for scalar-tensor theory

In scalar tensor theory we can consider a *generalised* harmonic gauge

$$0 = H^\mu \equiv g^{\nu\rho}\nabla_\nu\nabla_\rho x^\mu - J^\mu(g,\phi,\partial\phi)$$

For our scalar-tensor EFT (or a generic Horndeski theory) there is no choice for J^μ for which the gauge-fixed equations of motion are strongly hyperbolic in a generic weakly coupled background Papallo & HSR 2017, Papallo 2017

The problem can be seen at the linearised level and arises from the existence of unphysical solutions of the gauge-fixed equation. There are two types of unphysical mode:

- Pure gauge modes $h_{\mu\nu} \propto \nabla_{(\mu}\xi_{\nu)}$ where $\Box\xi_\mu = 0$
- Gauge condition violating modes: solutions with $H^\mu \neq 0$

A conventional (2-derivative) Einstein-scalar theory is strongly harmonic in harmonic gauge, so $M^i\xi_i$ is diagonalizable with real evals. Evals are degenerate (phase velocities of all modes are 1).

When we include the 4-derivative terms, at weak coupling we can treat these as a small deformation of $M^i\xi_i$. But a generic deformation of a matrix with degenerate evals is not diagonalizable!

The deformation causes non-trivial Jordan blocks to form. These are associated with subspaces spanned by the eigenvectors associated with pure-gauge and gauge-condition violating modes of the 2-derivative theory. Papallo & HSR 2017, Papallo 2017

The new idea: deform the gauge-fixing procedure to separates the speeds of the pure gauge and gauge condition violating modes in the 2-derivative theory. When we deform to a weakly coupled 4-derivative theory, $M^i\xi_i$ should remain diagonalisable.

GR in modified harmonic gauge

Introduce two auxiliary (inverse) metrics $\tilde{g}^{\mu\nu}$, $\hat{g}^{\mu\nu}$.
Modified gauge condition:

$$0 = H^\mu \equiv \tilde{g}^{\nu\rho}\nabla_\nu\nabla_\rho x^\mu = \tilde{g}^{\nu\rho}\Gamma^\mu_{\nu\rho}$$

Modified gauge-fixed equation:

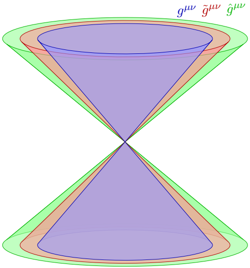
$$0 = G^{\mu\nu} + \hat{P}_\alpha{}^{\beta\mu\nu}\partial_\beta H^\alpha \qquad \hat{P}_\alpha{}^{\beta\mu\nu} \equiv \delta_\alpha^{(\mu}\hat{g}^{\nu)\beta} - \frac{1}{2}\delta_\alpha^\beta\hat{g}^{\mu\nu}$$

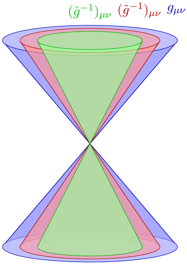
Implies $\hat{\Box} H^\mu + \ldots = 0$

In linearised theory, pure gauge solutions propagate along null cone of $\tilde{g}^{\mu\nu}$ and gauge-condition violating solutions propagate along null cone of $\hat{g}^{\mu\nu}$. Physical solutions propagate along null cone of $g^{\mu\nu}$.

The three metrics

We choose the unphysical metrics so that their null cones do not intersect each other or the null cone of the physical metric.





With this choice, can prove that GR is strongly hyperbolic in our modified harmonic gauge formulation.

Straightforward to include a minimally coupled (2-derivative) scalar field.

Well-posedness of scalar-tensor EFT

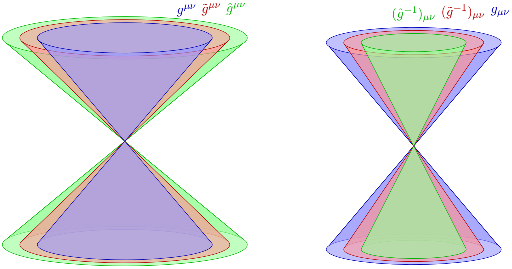
In the 2-derivative theory, $M^i \xi_i$ is diagonalisable with real evals.

The evals associated with pure gauge and gauge-condition violating modes are distinct from each other and from the evals associated with physical modes.

Using this, can show that $M^i \xi_i$ remains diagonalisable with real evals when we deform the theory to include 4-derivative terms, assuming weak coupling (i.e. a small deformation).

Hence, at weak coupling, our formulation gives strongly hyperbolic equations so the initial value problem is well posed.

Choosing the auxiliary metrics



One possibility:

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - a n^\mu n^\nu \qquad \hat{g}^{\mu\nu} = g^{\mu\nu} - b n^\mu n^\nu$$

where n^μ is unit normal to constant time slices and $0 < a < b$.

Numerics

The first numerical simulations using our formulation have been performed East & Ripley 2020:

- Shift-symmetric theory Einstein-scalar-GB

$$S = \int \frac{d^4x\sqrt{-g}}{16\pi G}\left(R - \frac{1}{2}(\partial\phi)^2 + \lambda\phi L_{\mathrm{GB}}\right)$$

- Dynamical scalarisation of rotating BHs
- Head-on collisions of BHs
- Inspiral and merger of BHs
- Typical values $\lambda M^{-2} \sim 0.01$ to 0.2

Generalisations

Our modified harmonic gauge formulation gives well-posed equations of motion for any weakly coupled Horndeski theory. Possible cosmological applications?

It also works for weakly coupled Lovelock theories such as Einstein-Gauss-Bonnet:

$$S = \int d^d x \sqrt{-g} \, (R + \alpha L_{GB})$$

Opens possibility of studying effect of higher curvature corrections on dynamical processes in $d > 4$ gravity e.g. black string instability?

Causality in gravitational theories (HSR 2021)

For the modified harmonic gauge Einstein (-scalar field) equation, the physical characteristics ξ_μ satisfy $g^{\mu\nu}\xi_\mu\xi_\nu = 0$. In a weakly coupled Lovelock/Horndeski theory, these characteristics lie on a cone that is close to the null cone of $g^{\mu\nu}$. Can we understand the properties of this characteristic cone in an *arbitrary* background?

Can discuss physical characteristics using an approach independent of any gauge-fixing procedure. (Christodoulou 2008).

Consider a general theory of gravity coupled to matter fields ϕ_I, with second order equations of motion, arising from a diffeomorphism invariant action

$$S = \frac{1}{16\pi G}\int d^d x\sqrt{-g}L(g,\phi_I)$$

Fix some (weakly coupled) background solution and consider equations linearised around it:

$$\begin{aligned} P_{gg}^{\mu\nu\rho\sigma\alpha\beta}\partial_\alpha\partial_\beta\delta g_{\rho\sigma} + P_{gm}^{\mu\nu I\alpha\beta}\partial_\alpha\partial_\beta\delta\phi_I + \ldots &= 0 \\ P_{mg}^{I\mu\nu\alpha\beta}\partial_\alpha\partial_\beta\delta g_{\mu\nu} + P_{mm}^{IJ\alpha\beta}\partial_\alpha\partial_\beta\delta\phi_J + \ldots &= 0 \end{aligned}$$

Tensors P_{gg} etc depend on background fields and their first and second derivatives. The *principal symbol* is

$$\mathcal{P}(\xi) = \begin{pmatrix} P_{gg}^{\mu\nu\rho\sigma\alpha\beta}\xi_\alpha\xi_\beta & P_{gm}^{\mu\nu I\alpha\beta}\xi_\alpha\xi_\beta \\ P_{mg}^{I\mu\nu\alpha\beta}\xi_\alpha\xi_\beta & P_{mm}^{IJ\alpha\beta}\xi_\alpha\xi_\beta \end{pmatrix}$$

This acts on "polarisation vectors" of form $(t_{\mu\nu}, t_I)$. But diffeomorphism symmetry implies that it can be regarded as acting on vectors $T \equiv ([t_{\mu\nu}], t_I)$ where $[t_{\mu\nu}]$ denotes an equivalence class w.r.t. $t_{\mu\nu} \sim t_{\mu\nu} + \xi_{(\mu}X_{\nu)}$. Such T corresponds to a "physical polarisation". Say that ξ_μ is a physical characteristic iff there exists $T \neq 0$ s.t. $\mathcal{P}(\xi)T = 0$.

Action principle and diffeomorphism invariance impose symmetries on principal symbol.

For $d = 4$ these imply that $P_{gg}^{\mu\nu\rho\sigma\alpha\beta}$ can be written in terms of an "effective metric" $C_{\mu\nu}$. For a weakly coupled theory this is close to $g_{\mu\nu}$.

Also $P_{gm}^{\mu\nu I\alpha\beta} = P_{mg}^{I\mu\nu\alpha\beta}$ and these can be written in terms of an object $C^{I\mu\nu\rho\sigma}$ with Riemann symmetries.

Can apply this formalism to Horndeski e.g. scalar-tensor EFT:

$$L = -V(\phi) + R + X + \alpha(\phi)X^2 + \beta(\phi)L_{\mathrm{GB}}$$

$$C_{\mu\nu} = g_{\mu\nu} - \beta'(\phi)\nabla_\mu\nabla_\nu\phi - \beta''(\phi)\nabla_\mu\phi\nabla_\nu\phi$$

$$C^{\mu\nu\rho\sigma} = -\beta'(\phi)\tilde{R}^{\mu\nu\rho\sigma}$$

Can show that ξ_μ is characteristic iff $p(\xi) = 0$ where the *characteristic polynomial* is

$$p(\xi) = (C^{-1})^{\mu\nu}\xi_\mu\xi_\nu Q(\xi)$$

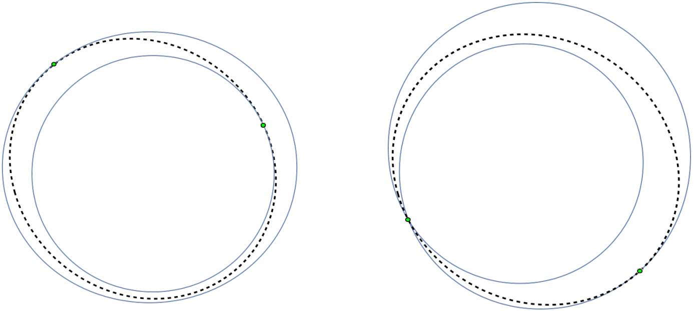
where Q is a homogeneous quartic polynomial with coefficients depending on the background fields and their first and second derivatives. p is a homogeneous polynomial of degree 6 which, remarkably, factorises into a product of quadratic and quartic polynomials. (Similar to linear elasticity theory for anisotropic solid with hexagonal symmetry e.g. Zinc.)

Note that degree 6 is the minimum degree for a system with 3 physical degrees of freedom.

In 2-derivative Einstein-scalar theory, $p(\xi) = -(g^{\mu\nu}\xi_\mu\xi_\nu)^3$. Factorisation also happens for other special theories or for symmetrical backgrounds e.g. FLRW.

Can show that the *characteristic cone* $p(\xi) = 0$ has 3 sheets at weak coupling. It is the union of quartic and quadratic cones.

Can visualise characteristic cone by taking a cross-section to define a "slowness surface" in $\mathbb{R}^3$. Quadratic cone always lies between sheets of quartic cone.



Left: slowness surface at generic point. Right: slowness surface at non-generic point.

Region inside quartic surface defines the *Gårding cone* in the cotangent space. The dual cone in the tangent space should be used to define notions of causality in these theories.

Summary

Effective field theory is an attractive formalism for parameterising possible strong field deviations from GR.

For the case of scalar-tensor theory, the leading EFT corrections have 4 derivatives.

For numerical simulations of BH mergers it is essential that a formulation of these equations is found that is strongly hyperbolic and hence admits a well-posed initial value problem.

We have found such a formulation, based on modified harmonic gauge. It is well-posed at weak coupling.

The first simulations of BH mergers have been performed using this formulation.

I have described a formalism for determining the physical characteristics of any gravitational theory with second order equations of motion.

For scalar-tensor EFT (or any Horndeski theory), in any background, the characteristic polynomial of degree 6 always factorises into a product of a quadratic and a quartic polynomial.

Future work: extend our results to other EFTs e.g. Einstein-Maxwell EFT.