A resurgent transseries for $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma

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Outline

1. Hydrodynamics of $\mathcal{N} = 4$ SYM plasma
2. A transseries for the energy density
3. The resurgent structure of the energy density
4. Checks of resurgence: predictions vs numerical calculations
5. Summary/Future Directions

Strongly coupled systems and dualities

Relativistic hydrodynamics: reliable description of strongly coupled systems, describing a slow evolution w.r.t. some microscopic scale

- real life: strongly coupled quark-gluon plasma seen in particle accelerators
- To determine the kinetic parameters of hydrodynamic equations (e.g. shear viscosity): study the associated microscopic theory

Associated microscopic description can can be a QFT, such as **strongly coupled $\mathcal{N} = 4$ Super Yang-Mills (SYM)**

- $N \to \infty$, gauge/gravity duality: determine hydrodynamic parameters, time dependent processes of the SYM plasma [Policastro *et al*,'01 - '04][Nastase,'05]

Strongly coupled systems and dualities

Interesting kinematic regime: expanding plasma in the so-called central rapidity region, where one can assume longitudinal boost invariance (Bjorken flow [Bjorken,'83])

- dynamics much simpler, but preserves complexity of the original problem

Analyse the time-dependent expansion of a boost invariant plasma system.

From AdS/CFT duality

- regularity of the dual geometry predicts almost perfect fluid hydrodynamic expansion [Janik, Peschanski,'05]
- leading dissipative corrections come from shear viscosity [Janik,'06]

Energy-momentum tensor

In hydrodynamic theories the energy-momentum tensor is given by

$$T^{\mu\nu} = \mathcal{E}\, u^{\mu} u^{\nu} + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^{\mu} u^{\nu}) + \Pi^{\mu\nu}$$

- $\mathcal{E}$ is energy density
  - $\Pi^{\mu\nu}$ is the shear stress tensor: dissipative effects
  - $\mathcal{P}(\mathcal{E}) = \mathcal{E}/3$ is pressure in $d = 4$ conformal theories
  - u is flow velocity - timelike eigenvector of the E-M tensor

Symmetries: conformal invariance, transversely homogeneous, invariance under longitudinal Lorentz boosts

Energy-momentum tensor

$$T^{\mu\nu} = \mathcal{E}\, u^\mu u^\nu + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

E-M conservation $\partial_\mu T^{\mu\nu} = 0$ and traceless $T^\mu_\mu = 0$

- dependence on single function: energy density in the local rest frame $\mathcal{E}$
- Boost invariance: $\mathcal{E}\,(\tau)$ only depends on proper time
- Conformal invariance: energy density and temperature related by power law

$$\mathcal{E} = \frac{3\pi^2 N^2}{8}\, T^4 \text{ for } \mathcal{N} = 4 \text{ SYM}$$

Strongly coupled SYM boost invariant plasma: **all physics encoded in $\mathcal{E}(\tau)$.**

Obtaining this function is in general too difficult:
perform a **large proper time expansion**

Late-time behaviour

- Starting from highly non-equilibrium initial conditions, the microscopic theory will reveal the transition to hydrodynamic behaviour at late times
- Conformal theories: late-time behaviour of temperature/energy density highly constrained

$$T(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}} \left(1 + \sum_{k=1}^{+\infty} \frac{t_k}{(\Lambda\tau)^{2k/3}}\right), \ \tau \gg 1,$$

  - Λ is a dimensionful parameter dependent on initial non-eq. conditions
  - Leading behaviour in $T(\tau)$ predicted by boost-invariant perfect fluid
  - subleading terms: dissipative hydrodynamic effects

AdS/CFT description of expanding plasma

- **Goal:** use dual geometry to analyse the expansion of boost invariant SYM plasma
- All possible behaviours of the spacetime expectation value of $\langle T_{\mu\nu} \rangle$
  - dependence only on single function $\mathcal{E}(\tau)$
- Dual geometry given by 5D metric [Hare *et at*,'00] [Skenderis,'02] [Fefferman,Graham,'85]

$$ds^2 = \frac{1}{z^2}\left(G_{\mu\nu} dx^\mu dx^\nu + dz^2\right),$$

  - z-"fifth coordinate"; $\mu = 0, \cdots, 3$
  - $G_{\mu\nu}\left(x^\alpha, z\right)$
- Solve Einstein equations with negative cosmological constant (asymptotic behaviour of geometry is AdS)

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - 6 G_{\mu\nu} = 0$$

AdS/CFT description of expanding plasma

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - 6 G_{\mu\nu} = 0$$

- Boundary condition at $z = 0$

$$G_{\mu\nu} = \eta_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \cdots$$

- Holographic renormalisation [Hare *et at*,'00]

$$\langle T_{\mu\nu} \rangle = \frac{N^2}{2\pi^2} g^{(4)}_{\mu\nu}$$

- Boost invariant plasma general metric:

$$ds^2 = \frac{1}{z^2} \left(dz^2 - \mathrm{e}^{-A} d\tau^2 + \tau^2 \mathrm{e}^{B} dy^2 + \mathrm{e}^{C} d\mathbf{x}_\perp^2 \right)$$

  - A, B, C functions of z, τ

- **Energy density** $$\mathcal{E}(\tau) = - \lim_{z \to 0} \frac{A(z, \tau)}{z^4}$$

AdS/CFT description of expanding plasma

- Equilibrium states of the microscopic theory (CFT) are represented by black hole solutions [Witten,'98]

flat space: planar horizons $\rightarrow$ black branes

- Simplest non-equilibrium phenomena to study: dynamics of linearised perturbations on top of the black brane

- perturbations of strongly coupled plasmas on top of equilibrium solution: black branes' Quasinormal modes (QNMs)

AdS dual geometry calculation

Use Eddington-Finkelstein coordinates:

$$ds^2 = -h\, r^2 d\tau^2 + 2d\tau\, dr + (r\tau + 1)^2\, \mathrm{e}^{b} dy^2 + r^2 \mathrm{e}^{-\frac{1}{2}(b+d)} d\mathbf{x}_\perp^2$$

- $0 \le r \le \infty$ holographic radial co-ord.; conformal boundary at $r \to \infty$
  - τ, y, $d\mathbf{x}_\perp^2$: reduce to proper time, rapidity and transverse co-ords
  - Define scaled variable $s = \frac{1}{r}\tau^{-1/3}$
- Minkowski metric at boundary $s = 0$

$$\lim_{s\to 0} h(s,\tau) = 1\,, \quad \lim_{s\to 0} b(s,\tau) = \lim_{s\to 0} d(s,\tau) = 0$$

- Horizon fixed at $s = 1$ to all orders

$$h(s = 1, \tau) = 0$$

- regular solutions in the bulk $0 < s < 1$

AdS dual geometry calculation

$$ds^2 = -h\, r^2 d\tau^2 + 2d\tau\, dr + (r\tau + 1)^2\, \mathrm{e}^b dy^2 + r^2 \mathrm{e}^{-\frac{1}{2}(b+d)} d\mathbf{x}_\perp^2$$

- Hydro expansion + transient modes: Transseries ansatz for $f = h, b, d$

$$f\left(s, u \equiv \tau^{2/3}, \boldsymbol{\sigma}\right) = \sum_{\mathbf{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\mathbf{n}} \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}\, u} \Phi_{\mathbf{n}}(u)$$

with power series expansions $\quad \Phi_{\mathbf{n}}(u) = u^{-\alpha_{\mathbf{n}}} \sum_{k\geq 0} f_k^{(\mathbf{n})}(s)\, u^{-k}$

- Determine large τ expansions of functions h, b, d requiring
  - Equilibrium given by black brane: $h_0^{(\mathbf{0})} = 1 - s^4$; $b_0^{(\mathbf{0})} = d_0^{(\mathbf{0})} = 0$
  - Flatness at boundary $s = 0$
  - Horizon at $s = 1$
  - Regularity in the bulk $0 < s < 1$
  - QMN frequencies: $\omega_k = -\frac{2\mathrm{i}}{3}\mathbf{e}_k \cdot \mathbf{A}$, $\mathbf{n} = \mathbf{e}_k = (0, \cdots, 0, 1, 0, \cdots)$

Retrieve the transseries expansion for the energy density $\mathcal{E}(\tau)$

Next

The energy density as a transseries

Energy density transseries for $u \equiv \tau^{2/3} \gg 1$

$$\mathcal{E}\left(u,\boldsymbol{\sigma}\right) = \sum_{\mathbf{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\mathbf{n}}\, \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}\,u}\,\Phi_{\mathbf{n}}\left(u\right) \ , \quad \Phi_{\mathbf{n}}\left(u\right) = u^{-\beta_{\mathbf{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{n})}\, u^{-k}$$

- infinite dimensional vector space $\mathbf{n} = \left(n_1, n_{\bar{1}}, n_2, n_{\bar{2}}, \cdots\right) \in \mathbb{N}_0^\infty$
- unit vectors $\mathbf{e}_k$, $\bar{\mathbf{e}}_k$

$$\mathbf{e}_1 \equiv (1,0,\cdots),\ \bar{\mathbf{e}}_1 \equiv (0,1,0,\cdots),\ \cdots$$

- Vector of "instanton" actions

$$\mathbf{A} = \left(A_1, \bar{A}_1, A_2, \bar{A}_2, \cdots\right)$$

- Transseries parameters

$$\boldsymbol{\sigma} = \left(\sigma_{A_1}, \sigma_{\bar{A}_1}, \sigma_{A_2}, \sigma_{\bar{A}_2}, \cdots\right)$$

such that $\boldsymbol{\sigma}^{\mathbf{n}} = \sigma_{A_1}^{n_1}\,\sigma_{\bar{A}_1}^{n_{\bar{1}}}\,\sigma_{A_2}^{n_2}\,\sigma_{\bar{A}_2}^{n_{\bar{2}}}\ \cdots$

$\Phi_{\mathbf{n}}\left(u\right)$ are asymptotic!

Energy density transseries

$$\mathcal{E}(u,\boldsymbol{\sigma}) = \sum_{\mathbf{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\mathbf{n}}\, \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}\,u}\,\Phi_{\mathbf{n}}(u)\ , \quad \Phi_{\mathbf{n}}(u) = u^{-\beta_{\mathbf{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{n})}\, u^{-k}$$

- Normalisations $\varepsilon_0^{(0)} = \pi^{-4}$; $\varepsilon_0^{(\mathbf{e}_k)} = 1,\ k \in \mathbb{N}$
- Hydrodynamic series

$$\Phi_0(u) = u^{-\beta_0} \sum_{k=0}^{+\infty} \varepsilon_k^{(0)}\, u^{-k},\ \beta_0 = 2$$

- Fundamental sectors: $\Phi_{\mathbf{n}}$ with $\mathbf{n} = \mathbf{e}_k, \overline{\mathbf{e}}_k$
  - Associated to QNM of the black brane
- Mixed sectors: $\Phi_{\mathbf{n}}$ with $\mathbf{n}$ a linear combination of different unit vectors
  - Associated to coupling of QNMs

Fundamental sectors

$$\mathcal{E}(u,\boldsymbol{\sigma}) = \sum_{\mathbf{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\mathbf{n}}\, \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}\,u}\, \Phi_{\mathbf{n}}(u)\ , \quad \Phi_{\mathbf{n}}(u) = u^{-\beta_{\mathbf{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{n})}\, u^{-k}$$

- Fundamental sector of lowest QNM $\omega_1 = 3.1195\cdots - \mathrm{i}\,2.7467\cdots$

$$\Phi_{\mathbf{e}_1}(u) = u^{-\beta_{\mathbf{e}_1}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{e}_1)}\, u^{-k}\ ;\ \Phi_{\bar{\mathbf{e}}_1}(u) = \overline{\Phi_{\mathbf{e}_1}(u)}$$

where $\beta_{\mathbf{e}_1} = -\frac{A_1}{6} + 3$ and $A_1 = \mathrm{i}\,\frac{3}{2}\,\omega_1$;

- Fundamental sector of QNM $\omega_2 = 5.16952\cdots - \mathrm{i}\,4.76357\cdots$:

$$\Phi_{\mathbf{e}_2}(u) = u^{-\beta_{\mathbf{e}_2}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{e}_2)}\, u^{-k}\ ;\ \Phi_{\bar{\mathbf{e}}_2}(u) = \overline{\Phi_{\mathbf{e}_2}(u)}$$

where $\beta_{\mathbf{e}_2} = -\frac{A_1}{6} + 3$ and $A_2 = \mathrm{i}\,\frac{3}{2}\,\omega_2$

Mixed sectors

$$\mathcal{E}(u,\boldsymbol{\sigma}) = \sum_{\mathbf{n}\in\mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\mathbf{n}}\, \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}\, u}\, \Phi_{\mathbf{n}}(u) \ , \quad \Phi_{\mathbf{n}}(u) = u^{-\beta_{\mathbf{n}}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{n})}\, u^{-k}$$

- Mixed sector associated to exponential weight $2A_1$

$$\Phi_{2\mathbf{e}_1}(u) = u^{-\beta_{2\mathbf{e}_1}} \sum_{k=0}^{+\infty} \varepsilon_k^{(2\mathbf{e}_1)}\, u^{-k}$$

with $\beta_{2\mathbf{e}_1} = 2\beta_{\mathbf{e}_1} - 2$

- Mixed sector associated to exponential weight $A_1 + \overline{A_1}$

$$\Phi_{\mathbf{e}_1+\bar{\mathbf{e}}_1}(u) = u^{-\beta_{\mathbf{e}_1+\bar{\mathbf{e}}_1}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{e}_1+\bar{\mathbf{e}}_1)}\, u^{-k}$$

with $\beta_{\mathbf{e}_1+\bar{\mathbf{e}}_1} = \beta_{\mathbf{e}_1} + \beta_{\bar{\mathbf{e}}_1} - 2$

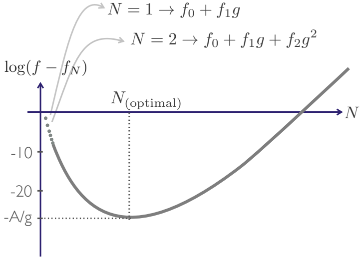
Next

All our sectors are asymptotic...

Aside: Asymptotic series

$$F(g) \simeq \sum_{n\geq 0} f_n\, g^n$$

- Divergent! No matter how small g is: $f_n\, g^n \to \infty$
- Truncate at some optimal $n = N$: very good approximation
- Take $g \ll 1$ fixed: define truncation $f_N(g) = \sum_{n=0}^{N} f_n\, g^n$



$N_{(\text{optimal})} \approx A/g$

Optimal error:

$(f - f_N)(g) \sim e^{-A/g}$

for some value A

Non-perturbative effect: $g \to 0$ invisible in perturbation theory!

Aside: Borel Transform & Resummation

Asymptotic series: $F(g) \simeq \sum_{n\geq 0} f_n\, g^{n+1}\,, \qquad \text{with} \quad F_n \sim n!$

- **Borel transform**: $\mathcal{B}[F](s) \;=\; \sum_{n=0}^{\infty} \frac{f_n}{n!}\, s^n$

  Rule: $\mathcal{B}\left[g^{\alpha+1}\right](s) = s^{\alpha}/\Gamma(\alpha+1)$

  - finite radius of convergence - find function $\mathcal{B}[F](s)$
  - In general $\mathcal{B}[F](s)$ will have singularities

- **Borel resummation** of F is the Laplace transform

$$\mathcal{S}F(g) = \int_0^{\infty} ds\, \mathcal{B}[F](s) \mathrm{e}^{-s/g}$$

Resurgence analysis and transseries [Écalle'81]

A transseries ($z \sim \infty$)

$$F(z,\sigma) = \sum_{n \geq 0} \sigma^n F^{(n)}, \qquad F^{(n)}(z) \simeq \mathrm{e}^{-nAz} \sum_{k \geq 0} F_k^{(n)} z^{-k}$$

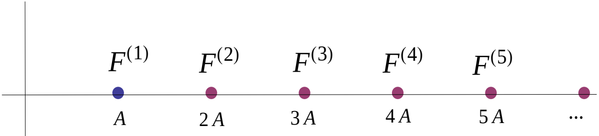
defines a resurgent function if it relates the asymptotics of multi- instanton contributions $F_n^{(\ell)}$ in terms of $F_n^{(\ell')}$ where ℓ' is close to ℓ

How does it work?

Resurgence at play[IA,Basar,Schiappa'18]

Multi-instanton asymptotic series

$$F(z) = \sum_{n=0}^{\infty} \sigma^n F^{(n)}(z)$$

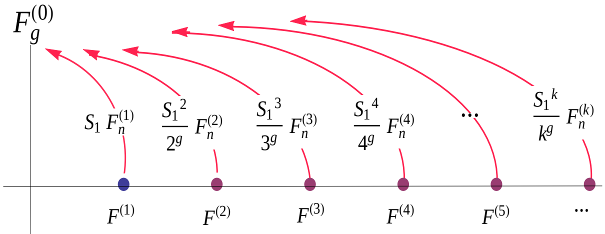


Perturbative series: $$F^{(0)}(z) = \sum_{g=0}^{\infty} F_g^{(0)} z^{-g-1}$$

Instanton series: $$F^{(n)}(z) = e^{-nAz} \sum_{g=1}^{\infty} F_g^{(n)} z^{-g}$$

Resurgence at play[IA,Basar,Schiappa'18]

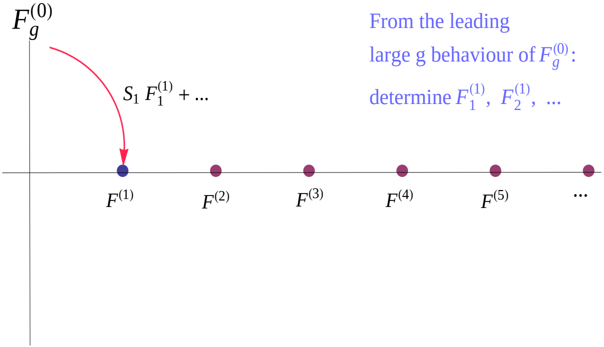
Large-order behaviour - Perturbative series for large g



All multi-instanton sectors contribute to the large-order behavior of coefficients F_g^0

$$F_g^{(0)} \sim S_1 \sum_{n>0} a_n(g)\, F_n^{(1)} + 2^{-g}\, S_1{}^2 \sum_{n>0} b_n(g)\, F_n^{(2)} + \ldots$$

Equivalently: Perturbative series for large g ENCODES all other sectors



From the leading
large g behaviour of $F_g^{(0)}$:
determine $F_1^{(1)},\ F_2^{(1)},\ \ldots$

$$F_g^{(0)} \sim S_1\left(F_1^{(1)} + \frac{A}{g-1}\, F_2^{(1)} + \ldots\right) + O(2^{-g})$$

Borel Analysis of the hydrodynamic series

Borel transform for the hydrodynamic expansion $\Phi_{\mathbf{0}}$

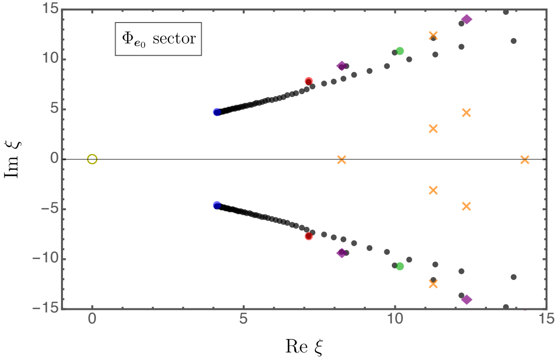
QNMs:

A_1; $2A_1$; $3A_1$

A_2;

A_3;

$\overline{A}_i$



$$A_1 = 3/2\,(2.746676 + 3.119452\mathrm{i});$$
$$A_2 = 3/2\,(4.763570 + 5.169521\mathrm{i});$$
$$A_3 = 3/2\,(6.769565 + 7.187931\mathrm{i});$$

How much information is encoded in the hydro series?

Resurgence predictions

- Complex-conjugate singularities in the Borel plane $\xi = A_1, \overline{A_1}, \cdots$
- Factorial growths $\Gamma\left(k+\beta\right)$ for $k \gg 1$, $\beta = \beta_{\mathbf{0}} - \beta_{\mathbf{e}_1} \in \mathbb{C}$
- Expected large-order behaviour:

$$\varepsilon_n^{(\mathbf{0})} \simeq - \frac{S_{\mathbf{0}\to\mathbf{e}_1}}{2\pi\mathrm{i}} \frac{\Gamma\left(n+\beta\right)}{A_1^{n+\beta}} \left(\varepsilon_0^{(\mathbf{e}_1)} + \frac{A_1}{n} \varepsilon_1^{(\mathbf{e}_1)} + \cdots \right) + \frac{\overline{S_{\mathbf{0}\to\mathbf{e}_1}}}{2\pi\mathrm{i}} \frac{\Gamma\left(n+\overline{\beta}\right)}{\overline{A}_1^{n+\overline{\beta}}} \left(\varepsilon_0^{(\overline{\mathbf{e}}_1)} + \frac{\overline{A}_1}{n} \varepsilon_1^{(\overline{\mathbf{e}}_1)} + \cdots \right) + \mathcal{O}\left(|A_2|^{-n}\right)$$

- Difficult to test with usual methods:
  - the (unknown) Borel residue mixes with β and A_1
  - oscillatory behaviour, convergence less obvious

Analyse asymptotic behaviour at the level of Borel singularities

Large order behaviour in the Borel plane

$$\varepsilon_n^{(\mathbf{0})} \simeq - \frac{S_{\mathbf{0}\to\mathbf{e}_1}}{2\pi\mathrm{i}} \frac{\Gamma\left(n+\beta\right)}{A_1^{n+\beta}} \left(\varepsilon_0^{(\mathbf{e}_1)} + \frac{A_1}{n}\varepsilon_1^{(\mathbf{e}_1)} + \cdots\right) + \frac{\overline{S_{\mathbf{0}\to\mathbf{e}_1}}}{2\pi\mathrm{i}} \frac{\Gamma\left(n+\overline{\beta}\right)}{\overline{A}_1^{n+\overline{\beta}}} \left(\varepsilon_0^{(\bar{\mathbf{e}}_1)} + \frac{\overline{A}_1}{n}\varepsilon_1^{(\bar{\mathbf{e}}_1)} + \cdots\right) + \mathcal{O}\left(|A_2|^{-n}\right)$$

- Growth associated to Borel singularity $\xi = A_1$: $\varepsilon_n^{(0)} \simeq \Gamma\left(n + \beta_{\mathbf{0}} - \beta_{\mathbf{e}_1}\right)$
- Multiply Φ_0 by $u^{-\alpha}$ s.t. Borel transform removes the *exact* factorial growth
- Behaviour at singular point is then $(\alpha = \beta_{\mathbf{e}_1})$

$$\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}}\Phi_{\mathbf{0}}\right](\xi) \sim S_{\mathbf{0}\to\mathbf{e}_1}\, \mathcal{B}\left[u^{\beta_{\mathbf{e}_1}}\Phi_{\mathbf{e}_1}\right](\xi - A_1)\, \frac{\log(\xi - A_1)}{2\pi\mathrm{i}} + \cdots$$

Analysing each branch cut of the Borel plane separately, we can **recover the coefficients** of the sector associated to that branch cut

Large order behaviour in the Borel plane

$$\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}}\Phi_{\mathbf{0}}\right](\xi) \sim S_{\mathbf{0}\to\mathbf{e}_1}\,\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}}\Phi_{\mathbf{e}_1}\right](\xi - A_1)\,\frac{\log(\xi - A_1)}{2\pi\mathrm{i}} + \cdots$$

- Transform the logarithmic behaviour into a square root branch cut:

$$\begin{aligned}\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}-1/2}\,\Phi_{\mathbf{0}}\right](\xi)\Big|_{\xi=A_1} &= \frac{S_{\mathbf{0}\to\mathbf{e}_1}}{2}\,\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}-1/2}\,\Phi_{\mathbf{e}_1}\right](\xi - A_1) + \cdots \\ &= \frac{S_{\mathbf{0}\to\mathbf{e}_1}}{2\sqrt{\xi - A_1}}\left(\frac{\varepsilon_0^{(\mathbf{e}_1)}}{\Gamma(1/2)} + \varepsilon_1^{(\mathbf{e}_1)}\,\frac{(\xi - A_1)}{\Gamma(3/2)} + \varepsilon_2^{(\mathbf{e}_1)}\,\frac{(\xi - A_1)^2}{\Gamma(5/2)} + \cdots\right)\end{aligned}$$

- Last step: transform it into a simple pole by defining $\xi = A_1 - (\zeta - A_1)^2$

$$\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}-1/2}\,\Phi_{\mathbf{0}}\right](\zeta)\Big|_{\zeta=A_1} = \frac{S_{\mathbf{0}\to\mathbf{e}_1}}{2\mathrm{i}\,(\zeta - A_1)}\left(\frac{\varepsilon_0^{(\mathbf{e}_1)}}{\Gamma(1/2)} - \varepsilon_1^{(\mathbf{e}_1)}\,\frac{(\zeta - A_1)^2}{\Gamma(3/2)} + \cdots\right)$$

Predictions of the leading fundamental sector

$$\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}-1/2}\,\Phi_{\mathbf{0}}\right](\zeta)\Big|_{\zeta=A_1} = \frac{\mathsf{S}_{\mathbf{0}\to\mathbf{e}_1}}{2\mathrm{i}\,(\zeta-A_1)}\left(\frac{\varepsilon_0^{(\mathbf{e}_1)}}{\Gamma(1/2)} - \varepsilon_1^{(\mathbf{e}_1)}\,\frac{(\zeta-A_1)^2}{\Gamma(3/2)} + \cdots\right)$$

- Residue at $\zeta = A_1$: determine $\mathsf{S}_{\mathbf{0}\to\mathbf{e}_1}$ $(\varepsilon_0^{(\mathbf{e}_1)} = 1)$
- Subtract the leading contribution:

$$\mathcal{B}\left[u^{\beta_{\mathbf{e}_1}-1/2}\,\Phi_{\mathbf{0}}\right](\zeta)\Big|_{\zeta=A_1} - \frac{\mathsf{S}_{\mathbf{0}\to\mathbf{e}_1}}{2\mathrm{i}\,(\zeta-A_1)}\,\frac{\varepsilon_0^{(\mathbf{e}_1)}}{\Gamma(1/2)} = -\varepsilon_1^{(\mathbf{e}_1)}\,\frac{\mathsf{S}_{\mathbf{0}\to\mathbf{e}_1}}{2\mathrm{i}\,\Gamma(3/2)}\,(\zeta-A_1) + \cdots$$

- Multiply by $(\zeta - A_1)^{-2}$ and take residue: prediction of $\varepsilon_1^{(\mathbf{e}_1)}$

Iterative process to obtain the coefficients of the leading fundamental sector $\Phi_{\mathbf{e}_1}$

Subleading singularities of perturbative series

- We analysed the properties of the leading branch cut (closest to the origin)
- Now analyse the behaviour at the second Borel singularity $\xi = A_2$
- How? Subtract the leading large-order behaviour at $\xi = A_1$ from the perturbative coefficients

$$\delta_1 \varepsilon_n^{(\mathbf{0})} \quad = \quad \varepsilon_n^{(\mathbf{0})} - \frac{S_{\mathbf{0}\to\mathbf{e}_1}}{2\pi\mathrm{i}} \frac{\Gamma\left(n+\beta\right)}{A_1^{n+\beta}} \chi_{\mathbf{0}\to\mathbf{e}_1}(n) - \mathrm{c.c} \simeq \mathcal{O}\left(|A_2|^{-n}\right)$$

- Contributions $\chi_{\mathbf{0}\to\mathbf{e}_1}(n)$ are asymptotic series in n^{-1}

$$\chi_{\mathbf{0}\to\mathbf{e}_1}(n) \simeq \left(\varepsilon_0^{(\mathbf{e}_1)} + \frac{A_1}{n}\varepsilon_1^{(\mathbf{e}_1)} + \cdots\right)$$

- Resum contribution for each value of n: Borel→Padé approx.→summation
- Define new series and analyse its Borel transform

$$\delta_1 \Phi_{\mathbf{0}}\left(u\right) = u^{-\beta_{\mathbf{0}}} \sum_k \delta_1 \varepsilon_k^{(\mathbf{0})}\, u^{-k}$$

Subleading singularities for hydro series

Borel transform for the subtracted hydrodynamic expansion

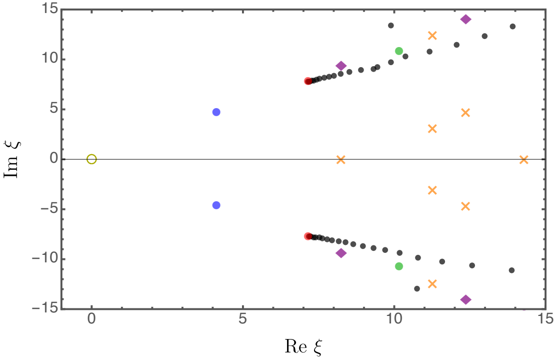
QNMs:

A_1; $2A_1$; $3A_1$

A_2;

A_3;

$\overline{A}_i$



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The leading singularities were effectively subtracted!

Predictions

With the procedures just described we can:

- Analyse the leading singularities of the hydro series and predict the coefficients associated to the fundamental sector $\Phi_{\mathbf{e}_1}$
- Subtract the contribution of the leading sectors and analyse the subleading singularities: predict the coefficients of the fundamental sector $\Phi_{\mathbf{e}_2}$
- This iterative procedure can be taken to reach extra subleading singularities
- Apply the procedure to fundamental sectors such as $\Phi_{\mathbf{e}_1}$: analyse its singularity structure, in particular contributions from the mixed sectors

Next

Compare resurgence predictions and gravity calculations

Resurgence predictions

- Predictions from the large order of hydro series $\Phi_{\mathbf{0}}$
  - Predicted coefficients of fundamental sectors $\Phi_{\mathbf{e}_1}$ and $\Phi_{\mathbf{e}_2}$
- Predictions from the large order of fundamental sector $\Phi_{\mathbf{e}_1}$
  - Predicted coefficients of mixed sectors $\Phi_{2\mathbf{e}_1}$ and $\Phi_{\mathbf{e}_1+\bar{\mathbf{e}}_1}$
- Determined respective Borel residues $S_{\mathbf{n}\to\mathbf{m}}$
- Compared results to gravity calculations:

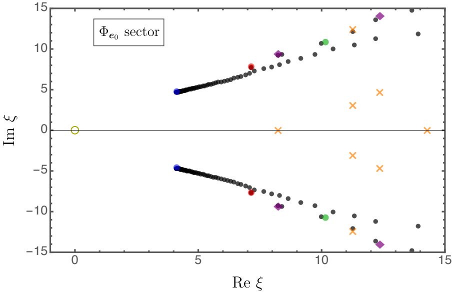
$$\Delta_{\mathbf{n}}\varepsilon_k^{(\mathbf{m})} \equiv \frac{\varepsilon_k^{(\mathbf{m})}\big|_{\mathbf{n}-\text{predicted}} - \varepsilon_k^{(\mathbf{m})}\big|_{\text{numerical}}}{\varepsilon_k^{(\mathbf{m})}\big|_{\text{numerical}}}\,,\ k \geq 1$$

  - $\varepsilon_k^{(\mathbf{m})}\big|_{\text{numerical}}$: coeffs of $\Phi_{\mathbf{m}}$ determined from gravity
  - $\varepsilon_k^{(\mathbf{m})}\big|_{\mathbf{n}-\text{predicted}}$: coeffs of $\Phi_{\mathbf{m}}$ predicted from the large order of sector $\Phi_{\mathbf{n}}$

Fundamental sector $\Phi_{\mathbf{e_1}}$ from hydro series

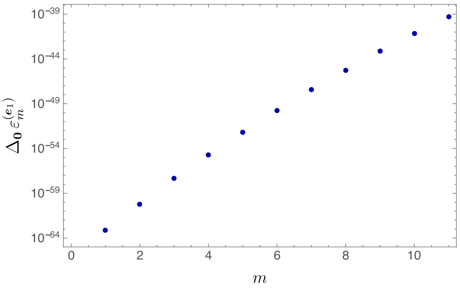
$$\Phi_{\mathbf{0}}(u) = u^{-\beta_0} \sum_{k=0}^{+\infty} \varepsilon_k^{(0)} u^{-k}, \; \beta_0 = 2, \; \text{(370 terms)}$$

Singularities of $\mathcal{B}[\Phi_0]$



Convergence of $\varepsilon_k^{(0)}$ to

first coefficients of $\Phi_{\mathbf{e_1}}$ sector

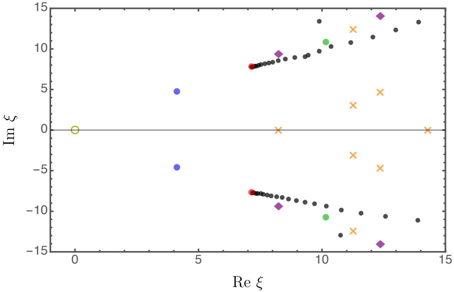


Borel residue: $S_{\mathbf{0}\to\mathbf{e_1}} = -0.01113168212 + 0.0305013486\mathrm{i}$

Fundamental sector $\Phi_{\mathbf{e}_2}$ from hydro series

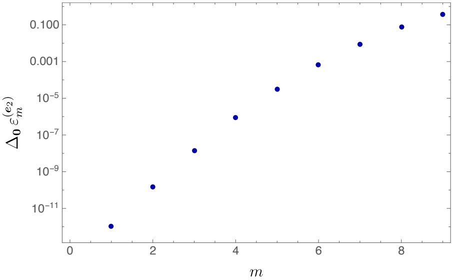
$$\delta_1 \Phi_{\mathbf{0}}(u) = u^{-\beta_0} \sum_{k=1}^{+\infty} \delta_1 \varepsilon_k^{(0)} u^{-k}, \ \beta_0 = 2, \ \text{(200 terms)}$$

Singularities of $\mathcal{B}[\delta_1 \Phi_0]$



Convergence of $\varepsilon_k^{(0)}$ to

first coefficients of $\Phi_{\mathbf{e}_2}$ sector

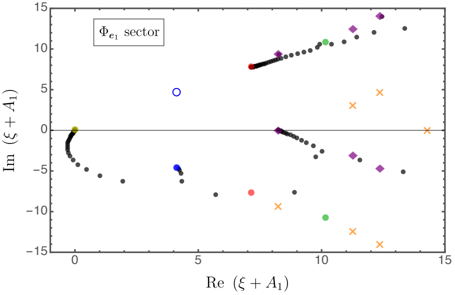


Borel residue: $S_{\mathbf{0}\to\mathbf{e}_2} = 0.170024383607 + 0.0974608479999\mathrm{i}$

Fundamental sector $\Phi_{\mathbf{e}_2}$ from sector $\Phi_{\mathbf{e}_1}$

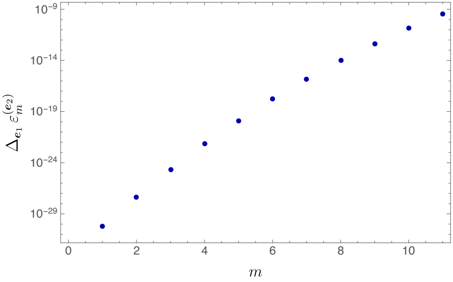
$$\Phi_{\mathbf{e}_1}(u) = u^{-\beta_{\mathbf{e}_1}} \sum_{k=0}^{+\infty} \varepsilon_k^{(\mathbf{e}_1)} u^{-k}, \quad \text{(270 terms)}$$

Singularities of $\mathcal{B}[\Phi_{\mathbf{e}_1}]$



Convergence of $\varepsilon_k^{(\mathbf{e}_1)}$ to

first coefficients of $\Phi_{\mathbf{e}_2}$ sector

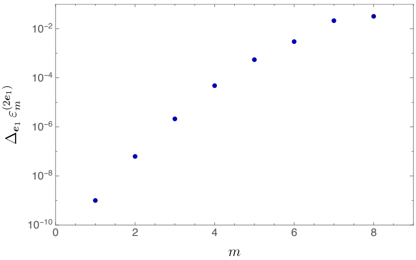


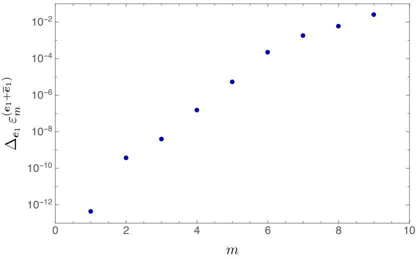
Borel residue: $S_{\mathbf{e}_1 \to \mathbf{e}_2} = 2.6127578014 - 10.6770578911\mathrm{i}$

Mixed sectors $\Phi_{2\mathbf{e}_1}$ and $\Phi_{\mathbf{e}_1+\bar{e}_1}$ from sector $\Phi_{\mathbf{e}_1}$

$$\delta_1\Phi_{\mathbf{e}_1}(u) = u^{-\beta_{\mathbf{e}_1}} \sum_{k=1}^{+\infty} \delta_1\varepsilon_k^{(\mathbf{e}_1)} u^{-k}, \text{ (200 terms)}$$

- Prediction of first coefficients of $\Phi_{2\mathbf{e}_1}$ and $\Phi_{\mathbf{e}_1+\bar{\mathbf{e}}_1}$ sectors from coefficients $\varepsilon_k^{(\mathbf{e}_1)}$





Summary & Future directions

Resurgent structure associated to SYM plasma undergoing Bjorken flow

- Calculation of a transseries for the energy density:
  - from the bulk dual geometry, using AdS/CFT duality
  - Exponentially suppressed sectors, associated to QNMs of a black brane
- Predictions from resurgence via residues of Borel transforms
  - Very accurate predictions, showing how all non-perturbative information is encoded in the hydro series
  - Method bypasses the intricate oscillatory behaviour of the large-order relations
  - Iterative process to obtain exponentially suppressed sectors from perturbative data

Summary & Future directions

- Coupling between QNMs
  - Appearance of mixed sectors, interpreted as non-trivial coupling between QNMs
  - Expected from resurgence but surprising from gravity as QNMs appear as solutions of linearised Einstein eqs
- Future directions:
  - Resummation and properties of the solution at early times, connection with attractor
  - Role of the residual initial conditions in transseries

Thank you!