Towards a theory of nonlinear gravitational waves: a systematic approach to nonlinear gravitational perturbations in vacuum

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based on [arXiv:1705.02258]

Southampton, 11th May 2017

Plan

1. Motivation
2. General approach to nonlinear perturbations
3. Some more general remarks
4. An illustrative example: gravitational perturbations of Schwarzschild–Minkowski/dS/AdS in static coordinates (see `[arXiv:1705.02258]` for a more complicated example: gravitational perturbations of de Sitter in comoving coordinates)
5. Applications: asymptotically AdS time-periodic solutions of vacuum Einstein equations (back to AdS)

Motivation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = T_{\alpha\beta}\,, \qquad \Lambda < 0$$

Anti-de Sitter (AdS) space: the maximally symmetric solution for the vacuum case; may be important due to AdS/CFT correspondence

Conjecture (Bizoń-R. 2011)

AdS_{d+1} (for $d \geq 3$) is unstable under arbitrarily small perturbations (against collapse)

Conjecture (Maliborski-R. 2013)

Negative cosmological constant allows for the existence of stable, globally regular, asymptotically AdS, time-periodic solutions of Einstein equations, immune to this instability

Evidence based on the study of a model case: Einstein equations sourced by massless, spherically symmetric scalar field (the same evidence for other matter fields at spherical symmetry; in vacuum case: only cohomogeneity-two biaxial Bianchi IX)

Urgent question: what happens outside spherical symmetry?

one possible way: to build perturbation expansion
(only to the 3rd order to check for instability with the resonant system (hard), in principle to arbitrarily high orders to construct time-periodic solutions (relatively easy and the 3rd order is probably crucial)).

1. Ó.J.C. Dias, G.T. Horowitz and J.E. Santos, *Gravitational Turbulent Instability of Anti-de Sitter Space* Class. Quant. Grav. **29**, 194002 (2012), `[arXiv:1109.1825]`
2. Ó.J.C. Dias and J.E. Santos, *AdS nonlinear instability: moving beyond spherical symmetry*, Class. Quantum Grav. **33** 23LT01 (2016) `[arXiv:1602.03890]`

but details of these works not given explicitly

3. R., *Higher order perturbations of Anti-de Sitter space and time-periodic solutions of vacuum Einstein equations*, `[arXiv:1701.07804]`

General approach to gravitational perturbations

The studies of gravitational perturbations of AdS led to a systematic and robust scheme for nonlinear gravitational perturbations in vacuum:

1. There are only two polarization states in gravitational waves: at each order of perturbation expansion there should exist two (for each gravitational mode) masters scalar variables satisfying an inhomogeneous linear wave equation with a uniquely defined potential (cf. [Regge&Wheeler, 57], [Zerilli, 70], [Nollert, 99])
2. At each order gauge invariant part of metric perturbations (like Regge-Wheeler gauge invariant variables) are uniquely given in terms of master scalar variables and their derivatives (and some source functions at nonlinear orders) (cf. [Mukohyama, 00], [Brizuela et al., 09])
3. These relations can be inverted for scalar master variables to be given in terms of RW type gauge invariant variables to provide the initial data and the form of scalar sources for the scalar wave equations for master scalar variables (cf. [Moncrief, 74], [Garat&Price, 00], [Brizuela et al., 09])

A few general remarks

1. Identities for the sources - crucial for the consistency of higher orders of perturbation expansion
2. Gauge issues can become a nuisance - we find fully gauge invariant approach to higher orders of perturbation expansion (cf. [Garat&Price, 00], [Brizuela et al., 09]) neither necessary nor useful
3. We use multipole expansion. At nonlinear orders of perturbation expansion the $\ell = 0, 1$ (monopole and dipole) parts need special treatment
4. We limit ourselves to axial symmetry (stepping beyond axial symmetry is a technical, not a conceptual issue). Then we can limit ourselves to polar perturbations only (including axial perturbations to the scheme is straightforward)
5. We illustrate our approach on concrete examples in given coordinate systems
6. Including matter - postponed to the future work

Perturbations in vacuum - general setup

Consider $R_{\mu\nu} - \kappa\frac{d}{\ell^2}g_{\mu\nu} = 0$ with $\kappa = 0, +1, -1$ and $\Lambda = \kappa\frac{d(d-1)}{2\ell^2}$.
Let $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ (in matrix notation $g = \bar{g} + \delta g$), then

$$\begin{aligned}
g^{\alpha\beta} &= \left(\bar{g}^{-1} - \bar{g}^{-1}\delta g\bar{g}^{-1} + \bar{g}^{-1}\delta g\bar{g}^{-1}\delta g\bar{g}^{-1} - \dots\right)^{\alpha\beta} = \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta}\,, \\
\Gamma^{\alpha}_{\mu\nu} &= \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}\left(\bar{g}^{-1} - \bar{g}^{-1}\delta g\bar{g}^{-1} + \bar{g}^{-1}\delta g\bar{g}^{-1}\delta g\bar{g}^{-1} - \dots\right)^{\alpha\lambda}\left(\bar{\nabla}_{\mu}\delta g_{\lambda\nu} + \bar{\nabla}_{\nu}\delta g_{\lambda\mu} - \bar{\nabla}_{\lambda}\delta g_{\mu\nu}\right) = \bar{\Gamma}^{\alpha}_{\mu\nu} + \delta\Gamma^{\alpha}_{\mu\nu}\,, \\
R_{\mu\nu} &= \bar{R}_{\mu\nu} + \bar{\nabla}_{\alpha}\delta\Gamma^{\alpha}_{\mu\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\alpha}_{\alpha\mu} + \delta\Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - \delta\Gamma^{\lambda}_{\mu\alpha}\delta\Gamma^{\alpha}_{\lambda\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}\,.
\end{aligned}$$

Now in Einstein equations $\delta R_{\mu\nu} - \kappa\frac{d}{\ell^2}\delta g_{\mu\nu} = 0$ expand $\delta g_{\mu\nu} = \sum_i \varepsilon^i h^{(i)}_{\mu\nu}$ itself and get the hierarchy of perturbative Einstein equations:

$$E^{(i)}_{\mu\nu} \equiv \Delta_L h^{(i)}_{\mu\nu} - S^{(i)}_{\mu\nu} = 0\,,$$

$$\Delta_L h_{\mu\nu} = \frac{1}{2}\left(-\bar{\nabla}^{\alpha}\bar{\nabla}_{\alpha}h_{\mu\nu} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h - 2\bar{R}_{\mu\alpha\nu\beta}h^{\alpha\beta} + \bar{\nabla}_{\mu}\bar{\nabla}^{\alpha}h_{\nu\alpha} + \bar{\nabla}_{\nu}\bar{\nabla}^{\alpha}h_{\mu\alpha}\right), \qquad h = \bar{g}^{\alpha\beta}h_{\alpha\beta},\ h^{\alpha\beta} = \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}h_{\mu\nu}$$

$$\begin{aligned}
S^{(i)}_{\mu\nu} = \left[\varepsilon^i\right]\Big\{ &-(1/2)\bar{\nabla}_{\alpha}\left[\left(-\bar{g}^{-1}\delta g\bar{g}^{-1} + \bar{g}^{-1}\delta g\bar{g}^{-1}\delta g\bar{g}^{-1} - \dots\right)^{\alpha\lambda}\left(\bar{\nabla}_{\mu}\delta g_{\lambda\nu} + \bar{\nabla}_{\nu}\delta g_{\lambda\mu} - \bar{\nabla}_{\lambda}\delta g_{\mu\nu}\right)\right] \\
&+(1/2)\bar{\nabla}_{\nu}\left[\left(-\bar{g}^{-1}\delta g\bar{g}^{-1} + \bar{g}^{-1}\delta g\bar{g}^{-1}\delta g\bar{g}^{-1} - \dots\right)^{\alpha\lambda}\left(\bar{\nabla}_{\mu}\delta g_{\lambda\alpha} + \bar{\nabla}_{\alpha}\delta g_{\lambda\mu} - \bar{\nabla}_{\lambda}\delta g_{\mu\alpha}\right)\right] - \delta\Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\mu\nu} + \delta\Gamma^{\lambda}_{\mu\alpha}\delta\Gamma^{\alpha}_{\lambda\nu}\Big\}
\end{aligned}$$

Spherical symmetry & Regge-Wheeler decomposition

transformation of tensor components under rotations

rotations -transformation of angular variables preserving $(\gamma_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$, $(\varepsilon_{ab}) = \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)

$$T_{\alpha\beta} = \begin{pmatrix} \boxed{S} & \boxed{S} & \boxed{V} \\ \boxed{S} & \boxed{S} & \boxed{V} \\ \boxed{V} & \boxed{V} & \boxed{T} \end{pmatrix}$$

$$S_{\ell m} = Y_{\ell m}(\theta,\phi), \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)}$$

$$\left(\overset{1}{V}_{\ell m}\right)_a = (S_{\ell m})_{;a}, \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)}$$

$$\left(\overset{2}{V}_{\ell m}\right)_a = \varepsilon_{ab}\gamma^{bc}(S_{\ell m})_{;c}, \quad \text{parity } (-1)^{\ell+1} \text{ (axial(or vector or odd) perturbation)}$$

$$\left(\overset{1}{T}_{\ell m}\right)_{ab} = (S_{\ell m})_{;a;b}, \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)}$$

$$\left(\overset{2}{T}_{\ell m}\right)_{ab} = \gamma_{ab}S_{\ell m}, \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)}$$

$$\left(\overset{3}{T}_{\ell m}\right)_{ab} = \varepsilon_{(ac}\gamma^{cd}(S_{\ell m})_{;d;b)}, \quad \text{parity } (-1)^{\ell+1} \text{ (axial(or vector or odd) perturbation)}$$

Polar perturbations at axial symmetry (on concrete example)

Schwarzschild in static coordinates: $ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega_2^2$, $\quad A = 1 - \kappa r^2/\ell^2 - 2M/r$, $\quad r^2A'' - 2A + 2 = 0$

$$h^{(i)}_{\alpha\beta} = \begin{pmatrix} h^{(i)}_{00} & h^{(i)}_{01} & h^{(i)}_{02} & 0 \\ h^{(i)}_{01} & h^{(i)}_{11} & h^{(i)}_{12} & 0 \\ h^{(i)}_{02} & h^{(i)}_{12} & h^{(i)}_{22} & 0 \\ 0 & 0 & 0 & h^{(i)}_{33} \end{pmatrix}$$

$$h^{(i)}_{00} = \sum_\ell h^{(i)}_{\ell\,00}(t,r)P_\ell(\cos\theta), \qquad h^{(i)}_{\ell\,00}(t,r) = \left(f^{(i)}_{\ell\,00} + 2\partial_t\zeta^{(i)}_{\ell\,0} - AA'\,\zeta^{(i)}_{\ell\,1}\right)$$

$$h^{(i)}_{11} = \sum_\ell h^{(i)}_{\ell\,11}(t,r)P_\ell(\cos\theta), \qquad h^{(i)}_{\ell\,11}(t,r) = \left(f^{(i)}_{\ell\,11} + 2\partial_r\zeta^{(i)}_{\ell\,1} + \frac{A'}{A}\,\zeta^{(i)}_{\ell\,1}\right)$$

$$h^{(i)}_{01} = \sum_\ell h^{(i)}_{\ell\,01}(t,r)P_\ell(\cos\theta), \qquad h^{(i)}_{\ell\,01}(t,r) = \left(f^{(i)}_{\ell\,01} + \partial_r\zeta^{(i)}_{\ell\,0} + \partial_t\zeta^{(i)}_{\ell\,1} - \frac{A'}{A}\,\zeta^{(i)}_{\ell\,0}\right)$$

$$h^{(i)}_{02} = \sum_\ell h^{(i)}_{\ell\,02}(t,r)\partial_\theta P_\ell(\cos\theta), \qquad h^{(i)}_{\ell\,02}(t,r) = \left(\zeta^{(i)}_{\ell\,0} + \partial_t\zeta^{(i)}_{\ell\,2}\right)$$

$$h^{(i)}_{12} = \sum_\ell h^{(i)}_{\ell\,12}(t,r)\partial_\theta P_\ell(\cos\theta), \qquad h^{(i)}_{\ell\,12}(t,r) = \left(\zeta^{(i)}_{\ell\,1} - \frac{2}{r}\zeta^{(i)}_{\ell\,2} + \partial_r\zeta^{(i)}_{\ell\,2}\right)$$

the sources $S^{(i)}_{\mu\nu}$ and

perturbation Einstein equations $E^{(i)}_{\mu\nu}$ are decomposed accordingly

$$\frac{1}{2}\left(h^{(i)}_{22} + \frac{h^{(i)}_{33}}{\sin^2\theta}\right) = \sum_\ell h^{(i)}_{\ell+}(t,r)P_\ell(\cos\theta), \qquad h^{(i)}_{\ell+}(t,r) = \left(r^2 f^{(i)}_{\ell+} + 2rA\zeta^{(i)}_{\ell\,1} - \ell(\ell+1)\zeta^{(i)}_{\ell\,2}\right),$$

$$\frac{1}{2}\left(h^{(i)}_{22} - \frac{h^{(i)}_{33}}{\sin^2\theta}\right) = \sum_\ell h^{(i)}_{\ell-}(t,r)\left(-\ell(\ell+1)P_\ell(\cos\theta) - 2\cot\theta\,\partial_\theta P_\ell(\cos\theta)\right), \qquad h^{(i)}_{\ell-}(t,r) = \zeta^{(i)}_{\ell\,2}$$

$f^{(i)}_{\ell\,00}(t,r), f^{(i)}_{\ell\,11}(t,r), f^{(i)}_{\ell\,01}(t,r), f^{(i)}_{\ell+}(t,r)$ are Regge-Wheeler (gauge invariant) variables

$\zeta^{(j)}_{\ell\,0}(t,r)$, $\zeta^{(j)}_{\ell\,1}(t,r)$, $\zeta^{(j)}_{\ell\,2}(t,r)$ define the j-th order polar gauge vector $\zeta^{(j)}_\alpha = \sum_\ell \left(\zeta^{(j)}_{\ell\,0}P_\ell(\cos\theta), \zeta^{(j)}_{\ell\,1}P_\ell(\cos\theta), \zeta^{(j)}_{\ell\,2}\partial_\theta P_\ell(\cos\theta), 0\right)$

and the corresponding gauge transformation $x^\mu \longrightarrow x^\mu + \varepsilon^j \zeta^{(j)\,\mu}$

$$\sum_{1\le i}\varepsilon^i h^{(i)}_{\mu\nu} \to \sum_{1\le i}\varepsilon^i h^{(i)}_{\mu\nu} + \varepsilon^j \mathscr{L}_{\zeta^{(j)}}\bar{g}_{\mu\nu} + \mathscr{O}\left(\varepsilon^{j+1}\right).$$

CAUTION: Regge-Wheeler gauge $\zeta^{(i)}_{\ell\,\mu} = 0$ is not asymptotically AdS (nor asymptotically flat in corresponding flat case)!

General approach to gravitational perturbations (2)

$$E^{(i)}_{\mu\nu} \equiv \Delta_L h^{(i)}_{\mu\nu} - S^{(i)}_{\mu\nu} = 0$$

- The *Lorentzian* Lichnerowicz operator $\Delta_L h^{(i)}_{\mu\nu}$ contains only RW gauge invariant variables $f^{(i)}_{\ell\,00}, f^{(i)}_{\ell\,11}, f^{(i)}_{\ell\,01}, f^{(i)}_{\ell\,+}$ and their derivatives
- The sources $S^{(i)}_{\mu\nu}$ depend on gauge choices made at lower orders, i.e. $\zeta^{(j)}_{\alpha}, j < i$

Nevertheless $E^{(i)}_{\mu\nu}$ are still a mess!

However at each order there should exist only **one scalar gravitational degree of freedom - a master scalar variable** (for polar/axial perturbations, and for a given multipole ℓ), satisfying some (in)homogeneous linear wave equation, to rule all $E^{(i)}_{\mu\nu}$

There are only two polarization in gravitational wave after all!

Problems to be fixed

- to identify/find the correct definition of master scalar variables at higher orders
- to find the source terms for the (inhomogeneous) wave equations for these scalar variables from the sources $S^{(i)}_{\mu\nu}$
- to switch between the metric perturbations and scalar variables easily
- to deal with the special cases $\ell = 0$ and $\ell = 1$ (these are pure gauges at linear order)
- to set the metric perturbations to asymptotically desired form with a suitable gauge transformation

General approach to gravitational perturbations (3)

1. At each order there is only **one scalar gravitational degree of freedom** (for polar/axial perturbations, and for a given multipole ℓ) satisfying (in)homogeneous linear wave equation with a potential (to be determined)

$$\tilde{\Box}_\ell \Phi^{(i)}_{\mathcal{P}\,\ell}(t,r) := r\left(-\bar{\Box} + V_\ell\right)\frac{\Phi^{(i)}_{\mathcal{P}\,\ell}(t,r)}{r} = \tilde{S}^{(i)}_{\mathcal{P}\,\ell} \tag{1}$$

2. RW variables $f^{(i)}_{\ell\,+}, f^{(i)}_{\ell\,11}, f^{(i)}_{\ell\,01}, f^{(i)}_{\ell\,00}$ are given as linear combinations of $\Phi^{(i)}_{\mathcal{P}\,\ell}$ and its derivatives (+ source functions at nonlinear orders):

$$f^{(i)}_{\ell\,+} = B\Phi^{(i)}_{\mathcal{P}\,\ell} + C\partial_t\Phi^{(i)}_{\mathcal{P}\,\ell} + D\partial_r\Phi^{(i)}_{\mathcal{P}\,\ell} + E\partial_{tr}\Phi^{(i)}_{\mathcal{P}\,\ell} + F\partial_{rr}\Phi^{(i)}_{\mathcal{P}\,\ell} + \alpha^{(i)}_\ell(t,r)\,, \tag{2}$$

$$f^{(i)}_{\ell\,11} = \cdots + \beta^{(i)}_\ell(t,r)\,, \qquad f^{(i)}_{\ell\,01} = \cdots + \gamma^{(i)}_\ell(t,r)$$

3. Satisfying (perturbative) Einstein equations fixes the potential V_ℓ and the coefficient functions in the equations above **uniquely (!)**
4. The relations (2) can be inverted for $\Phi^{(i)}_{\mathcal{P}\,\ell}$. There is a **unique (!)** way compatible with the ADM initial problem formulation. This also gives the source $\tilde{S}^{(i)}_{\mathcal{P}\,\ell}$ in (1) **uniquely (!)**

Trivial technicalities

In fact

$$E^{(i)}_{\ell\,-} = \frac{1}{4}\left(A^{-1} f^{(i)}_{\ell\,00} - A f^{(i)}_{\ell\,11}\right) - S^{(i)}_{\ell\,-}$$

and

$$0 = \frac{1}{2}\left(A E^{(i)}_{\ell\,11} - A^{-1} E^{(i)}_{\ell\,00}\right) + A^{-1}\partial_t E^{(i)}_{\ell\,02} - A\partial_r E^{(i)}_{\ell\,12} - \frac{2A + rA'}{r} E^{(i)}_{\ell\,12}$$
$$+ \frac{(\ell-1)(\ell+2)}{r^2} E^{(i)}_{\ell\,-}$$

sets an identity for the sources

Identities for the sources $S_{\mu\nu}^{(i)}$ - crucial for the consistency of higher orders of perturbation expansion

Taking the background divergence of perturbation Einstein equations

$$\bar{\nabla}^{\mu} E_{\mu\nu}^{(i)} \equiv 0$$

gives identities for the sources $S_{\mu\nu}^{(i)}$ (!)

$$\frac{1}{2}\left(\frac{1}{A}\partial_t S_{\ell\,00}^{(i)} + A\partial_t S_{\ell\,11}^{(i)}\right) + \frac{1}{r^2}\partial_t S_{\ell\,+}^{(i)} - A\partial_r S_{\ell\,01}^{(i)} - \frac{2A+rA'}{r} S_{\ell\,01}^{(i)} + \frac{\ell(\ell+1)}{r^2} S_{\ell\,02}^{(i)} \equiv 0\,,$$

$$\frac{1}{2}\left(\frac{1}{A}\partial_r S_{\ell\,00}^{(i)} + A\partial_r S_{\ell\,11}^{(i)}\right) - \frac{1}{r^2}\partial_r S_{\ell\,+}^{(i)} - \frac{1}{A}\partial_t S_{\ell\,01}^{(i)} + \frac{2A+rA'}{r} S_{\ell\,11}^{(i)} - \frac{\ell(\ell+1)}{r^2} S_{\ell\,12}^{(i)} \equiv 0\,,$$

$$\frac{1}{2}\left(\frac{1}{A} S_{\ell\,00}^{(i)} - A S_{\ell\,11}^{(i)}\right) - \frac{1}{A}\partial_t S_{\ell\,02}^{(i)} + A\partial_r S_{\ell\,12}^{(i)} + \frac{2A+rA'}{r} S_{\ell\,12}^{(i)} - \frac{(\ell-1)(\ell+2)}{r^2} S_{\ell\,-}^{(i)} \equiv 0\,.$$

Perturbations of spherically symmetric spaces, $A = 1 + \kappa r^2/\ell^2 - 2M/r$ (an easy way to the Zerilli equation)

master wave equation:

$$\tilde{\Box}_\ell \Phi^{(i)}_{\mathcal{P}\,\ell} := \frac{1}{A}\partial_{tt}\Phi^{(i)}_{\mathcal{P}\,\ell} - A\,\partial_{rr}\Phi^{(i)}_{\mathcal{P}\,\ell} - A'\,\partial_r\Phi^{(i)}_{\mathcal{P}\,\ell} + \left(\frac{A'}{r} + V_\ell\right)\Phi^{(i)}_{\mathcal{P}\,\ell} = \tilde{S}^{(i)}_{\mathcal{P}\,\ell}$$

potential (the celebrated Zerilli potential in the Schwarzschild case):

$$V_\ell = \frac{\ell(\ell+1)}{r^2} - \frac{A'}{r} + \underbrace{\left(2A - rA' - 2\right)}_{-6M/r}\frac{2A\,(rA'-2) - (rA')^2 + \ell^2(\ell+1)^2}{r^2\,(2A - rA' - \ell(\ell+1))^2}$$

and RW variables in terms of the master scalar variable (and source functions at nonlinear orders):

$$f^{(i)}_{\ell\,+} = A\partial_r\Phi^{(i)}_{\mathcal{P}\,\ell} + \frac{1}{r}\left(\frac{\ell(\ell+1)}{2} - \frac{2A - rA' - 2}{2A - rA' - \ell(\ell+1)}A\right)\partial_t\Phi^{(i)}_{\mathcal{P}\,\ell} + \alpha^{(i)}_\ell(t,r)$$

$$f^{(i)}_{\ell\,11} = \cdots + \beta^{(i)}_\ell(t,r)$$

$$f^{(i)}_{\ell\,01} = \cdots + \gamma^{(i)}_\ell(t,r)$$

Perturbations of spherically symmetric spaces, $A = 1 + \kappa r^2/\ell^2 - 2M/r$

$$\tilde{\Box}_\ell \Phi^{(i)}_{\mathcal{P}\,\ell} := \frac{1}{A}\partial_{tt}\Phi^{(i)}_{\mathcal{P}\,\ell} - A\,\partial_{rr}\Phi^{(i)}_{\mathcal{P}\,\ell} - A'\,\partial_r\Phi^{(i)}_{\mathcal{P}\,\ell} + \left(\frac{A'}{r} + V_\ell\right)\Phi^{(i)}_{\mathcal{P}\,\ell} = \tilde{S}^{(i)}_{\mathcal{P}\,\ell}$$

The master variable in terms of RW potentials - the **unique** form compatible with the ADM initial problem formulation:

$$\Phi^{(i)}_{\mathcal{P}\,\ell} = \frac{2r}{\ell(\ell+1)}\left(f^{(i)}_{\ell\,Y} + 2A\frac{Af^{(i)}_{\ell\,11} - r\partial_r f^{(i)}_{\ell\,Y}}{\ell(\ell+1) - 2A + rA'}\right)$$

its source at higher orders can be read off accordingly

$$\begin{aligned}
\tilde{S}^{(i)}_{\mathcal{P}\,\ell} = {} & \frac{4r^2}{(\ell-1)\ell(\ell+1)(\ell+2)}\left(\frac{A}{r}\left(A\,{}^{(i)}S_{\ell\,11} - \frac{1}{A}\,{}^{(i)}S_{\ell\,00}\right) + \frac{(\ell-1)(\ell+2) - 2\,(3A-2)}{r^3}\,{}^{(i)}S_{\ell\,+} - 2A\partial_r\left({}^{(i)}S_{\ell\,+}\Big/r^2\right)\right.\\
& - \frac{2\ell(\ell+1)}{r^2}A\,{}^{(i)}S_{\ell\,12} + \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{r^3}\,{}^{(i)}S_{\ell\,-}\\
& - \frac{2A - rA' - 2}{2A - rA' - \ell(\ell+1)}\left(\frac{A}{r}\left(A^{(i)}S_{\ell\,11} - \frac{1}{A}\,{}^{(i)}S_{\ell\,00}\right) - \frac{A(\ell-1)(\ell+2)}{r\,(2A - rA' - \ell(\ell+1))}\left(A^{(i)}S_{\ell\,11} + \frac{1}{A}\,{}^{(i)}S_{\ell\,00}\right)\right.\\
& \left.\left. -2\,\frac{3A\,(2A - rA' - 2) - \ell(\ell+1)\,(2A - rA' - \ell(\ell+1)) - 2(\ell-1)(\ell+1)A}{r^3\,(2A - rA' - \ell(\ell+1))}\,{}^{(i)}S_{\ell\,+} - 2A\partial_r\left({}^{(i)}S_{\ell\,+}\Big/r^2\right) - \frac{2\ell(\ell+1)}{r^2}A^{(i)}S_{\ell\,12}\right)\right)
\end{aligned}$$

To fix the source functions $\alpha_\ell^{(i)}$, $\beta_\ell^{(i)}$ and $\gamma_\ell^{(i)}$ we write them down as linear combinations of the sources $S^{(i)}_{\ell\,\mu\nu}$ and their first derivatives. Fixing $3\times 7\times 3=63$ function coefficients of these linear combinations is a technical task. It turns out that 54 functions (out of 63) are fixed in terms of 9 free functions. Moreover, in the resulting expressions, coefficients of these 9 free functions are identically zero due to the identities for the sources, thus the final expressions are **uniquely** defined:

$$\alpha_\ell^{(i)} = -\frac{2A\left(r^2\left(A^{-1}S^{(i)}_{\ell\,00} - AS^{(i)}_{\ell\,11}\right) + 2S^{(i)}_{\ell\,+}\right)}{\ell(\ell+1)\left(\ell(\ell+1)-2A+rA'\right)}$$

$$\beta_\ell^{(i)} = \frac{1}{A}\left(r\partial_r\alpha_\ell^{(i)} - \frac{\ell(\ell+1)-2A+rA'}{2A}\alpha_\ell^{(i)}\right)$$

$$\gamma_\ell^{(i)} = -\frac{2r\left(r^2\left(A^{-1}\partial_t S^{(i)}_{\ell\,00} + A\partial_t S^{(i)}_{\ell\,11}\right) + 2\partial_t S^{(i)}_{\ell\,+} - r\left(\ell(\ell+1)-2A+rA'\right)S^{(i)}_{\ell\,01}\right)}{\ell(\ell+1)\left(\ell(\ell+1)-2A+rA'\right)}$$

Back to AdS - gauge issues

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ is asymptotically AdS iff the Killing equation is satisfied in asymptotic sense:

$$\mathscr{L}_\xi g_{\mu\nu} = \mathscr{O}\left(\delta g_{\mu\nu}\right)$$

This puts the following falloff conditions [Bantilan, Pretorius & Gubser, 2012]: $\delta g_{\mu\nu} \sim \mathscr{O}\left(1/r^{\gamma_{\mu\nu}}\right)$, with $\gamma_{rr} = 5$, $\gamma_{rv} = 4$ and $\gamma_{\mu\nu} = 1$ (in $3+1$ dimensions).

For $\frac{1}{A}\partial_{tt}\Phi^{(i)}_{\ell} - A\,\partial_{rr}\Phi^{(i)}_{\mathcal{P}\,\ell} - A'\,\partial_r\Phi^{(i)}_{\mathcal{P}\,\ell} + \frac{\ell(\ell+1)}{r^2}\,\Phi^{(i)}_{\mathcal{P}\,\ell} = 0$ we have

$$\Phi^{(i)}_{\mathcal{P}\,\ell} \sim \alpha_\ell + \frac{\beta_\ell}{r} + \mathscr{O}\left(\frac{1}{r^2}\right);$$

for polar perturbations the necessary condition reads $\beta_\ell = 0$ [Dias, Horowitz & Santos, 2011]. Still RW gauge is not asymptotically AdS and suitable gauge transformation is needed

$$\zeta^{(i)}_{\ell\,1} = \frac{1}{4}\left(l^2 A^{-1}\partial_{tt}\Phi^{(i)}_{\mathcal{P}\,\ell} - 4r\partial_r\Phi^{(i)}_{\mathcal{P}\,\ell} - r^2\partial_{rr}\Phi^{(i)}_{\mathcal{P}\,\ell}\right),$$

$$\zeta^{(i)}_{\ell\,0} = -r\partial_t\zeta^{(i)}_{\ell\,1} + \frac{r}{3A}\partial_t\left(l^2\partial_{tt}\Phi^{(i)}_{\mathcal{P}\,\ell} + \Phi^{(i)}_{\mathcal{P}\,\ell}\right),$$

$$\zeta^{(i)}_{\ell\,2} = \frac{r}{3}\zeta^{(i)}_{\ell\,1}.$$

$\ell = 0$ special case

the only nontrivial equations are

$0 = E^{(i)}_{0\,+} = E^{(i)}_{0\,01} = E^{(i)}_{0\,1\pm0} := \left(AE^{(i)}_{0\,11} \pm A^{-1}E^{(i)}_{0\,00}\right)$. In fact

$$E^{(i)}_{0\,1-0} - \frac{1}{r}\partial_r E^{(i)}_{0\,+} + \frac{r}{2}\partial_r E^{(i)}_{0\,1+0} - rA^{-1}\partial_t E^{(i)}_{0\,01} + A^{-1}\left(1+2\frac{r^2}{\ell^2}\right)E^{(i)}_{0\,1+0} \equiv 0,$$

$$\partial_t\left(E^{(i)}_{0\,+} + \frac{r^2}{2}E^{(i)}_{0\,1+0}\right) - 2r\left(1+2\frac{r^2}{\ell^2}\right)E^{(i)}_{0\,0,1} - r^2 A\partial_r E^{(i)}_{0\,0,1} \equiv 0$$

and $f^{(i)}_{0\,01}, f^{(i)}_{0\,Y}$ become gauge degrees of freedom. Thus we are left with two equations $0 = E^{(i)}_{0\,0,1} = E^{(i)}_{0\,1+0}$ for two unknown functions $f^{(i)}_{0\,00}$ and $f^{(i)}_{0\,11}$, where $f^{(i)}_{0\,01}$ and $f^{(i)}_{0\,Y}$ can be freely specified and we put them to zero. This system can be easily integrated to yield

$$f^{(i)}_{0\,11} = f^{(i)}_{res\,0} + rA^{-1}\int S^{(i)}_{0\,01}\,dt$$

$$f^{(i)}_{0\,00} = A^2 f^{(i)}_{0\,11} - A\int r\left(S^{(i)}_{0\,11} + A^{-2}S^{(i)}_{0\,00}\right)dr\,,$$

with the residual degree of freedom $f^{(i)}_{res\,0} \equiv f^{(i)}_{res\,0}(r)$ that is not set by the equations $0 = E^{(i)}_{0\,0,1} = E^{(i)}_{0\,1+0}$. It is however uniquely determined as the solution of the first order ordinary differential equation set by (the time independent part of) the equation $E^{(i)}_{0\,+} = 0$.

Application: aAdS, regular, time-periodic solutions of vacuum Einstein equations (geons)

$\Box\Phi_{\mathcal{P}\,\ell} := A^{-1}\partial_{tt}\Phi_{\mathcal{P}\,\ell} - A\,\partial_{rr}\Phi_{\mathcal{P}\,\ell} - A'\,\partial_r\Phi_{\mathcal{P}\,\ell} + \frac{\ell(\ell+1)}{r^2}\,\Phi_{\mathcal{P}\,\ell} = 0$ and boundary conditions: regularity at $r=0$ and $\Phi_{\mathcal{P}\,\ell} \sim 1 + \mathcal{O}\left(1/r^2\right)$ give $\Phi_{\mathcal{P}\,\ell}(t,r) = \sum_j A_{\ell,j} e_{\ell,j}(r)\cos\left(\omega_{\ell,j}\, t/\ell + B_{\ell,j}\right)$ with $\omega_{\ell,j} = 1+\ell+2j$ and $e_{\ell,j}(r) = \frac{\ell\, r^{\ell+1}}{\left(1+r^2/\ell^2\right)^{\frac{\ell+1}{2}}}\;{}_2F_1\left(-j, 1+\ell+j; \tfrac{1}{2}; \frac{1}{1+r^2/\ell^2}\right)$

Ó.J.C. Dias and J.E. Santos, *AdS nonlinear instability: moving beyond spherical symmetry*, Class. Quantum Grav. **33** 23LT01 (2016) [arXiv:1602.03890]

Normal mode $\{\ell,m,p,\bar\omega\}$ at $\mathcal{O}(\varepsilon)$	# modes $\mathcal{O}\left(\varepsilon^2\right)$	# modes $\mathcal{O}\left(\varepsilon^3\right)$	Removable resonance $\left(-L\,\omega^{(2)}\right)$	Secular resonances $\{\ell,m,p,\omega\}$
$\{\mathbf{2},\mathbf{0},\mathbf{0},\,\frac{\mathbf{3}}{\mathbf{L}}\}_{\mathbf{s}}$	$6_{\mathbf{s}}$ $0_{\mathbf{v}}$	$8_{\mathbf{s}}$ $0_{\mathbf{v}}$	$\{2,0,0,\,\frac{3}{L}\}_{\mathbf{s}}$ $\left(\frac{3663}{8960}\right)$	None **(Geon ?)**
$\{2,0,1,\,\frac{5}{L}\}_{\mathbf{s}}$	$6_{\mathbf{s}}$ $0_{\mathbf{v}}$	$8_{\mathbf{s}}$ $0_{\mathbf{v}}$	$\{2,0,1,\,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{34397}{5376}\right)$	$\{4,0,0,\,\frac{5}{L}\}_{\mathbf{s}}$
$\{4,0,0,\,\frac{5}{L}\}_{\mathbf{s}}$	$10_{\mathbf{s}}$ $0_{\mathbf{v}}$	$14_{\mathbf{s}}$ $0_{\mathbf{v}}$	$\{4,0,0,\,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{52311625}{21446656}\right)$	$\{2,0,1,\,\frac{5}{L}\}_{\mathbf{s}}$

This is a purely technical obstruction due to the degeneracy of the spectrum! There is a geon bifurcating from **each** linear eigenfrequency. Take

$$\Phi_{\mathcal{P}}(t,r,\theta) = \varepsilon\,\Phi^{(1)}_{\mathcal{P}}(t,r,\theta) + \varepsilon^2\,\Phi^{(2)}_{\mathcal{P}}(t,r,\theta) + \varepsilon^3\,\Phi^{(3)}_{\mathcal{P}}(t,r,\theta) + \mathcal{O}\left(\varepsilon^4\right)$$

with a linear combination of two eigenmodes with $\omega = 5$ as the seed:

$$\Phi^{(1)}_{\mathcal{P}}(t,r,\theta) = \left(\eta\, e_{2,1}(r)P_2(\cos\theta) + (1-\eta)\,e_{4,0}(r)P_4(\cos\theta)\right)\cos\left((5+\varepsilon^2\omega_2)t/\ell\right).$$

the resonances can be removed iff

$$-651980329\,\eta^3 + 673396185\,\eta^2 - 358711575\,\eta + 22494375 = 49201152\,\eta\,\omega_2$$

$$16847182891\,\eta^3 - 38330631185\,\eta^2 + 31825994625\,\eta - 10200766875 = 4182097920\,(1-\eta)\,\omega_2\,.$$

Secular terms

third order: $\Box_\ell \Phi_\ell^{(3)} = \tilde{S}_\ell^{(3)}$; projection on the basis $\{e_{\ell,j}\}$ gives an infinite set of decoupled forced harmonic oscillations for the generalized Fourier coefficients $c_{\ell,j}(t) := (e_{\ell,j}, \Phi_\ell)$:

$$\ddot{c}_{\ell,j} + \omega_{\ell,j}^2 c_{\ell,j} = \tilde{F}_{\ell,j}^{(3)} := (e_{\ell,j}, A\,\tilde{S}_\ell^{(3)})\,.$$

Then, in general, secular terms arise:

$$\ddot{g}(t) + \omega_0^2\, g(t) = a\cos(\omega t)\,,$$

$$g(t) = \frac{\dot{g}(0)}{\omega_0}\sin(\omega_0 t) + g(0)\cos(\omega_0 t) + \begin{cases} \dfrac{a\left(\cos(\omega t) - \cos(\omega_0 t)\right)}{\omega_0^2 - \omega^2}\,, & \omega_0 \neq \omega\,, \\[2ex] \dfrac{a}{2\omega_0}\, t\sin(\omega_0 t)\,, & \omega_0 = \omega\,. \end{cases}$$

Closing remarks in the AdS context

- The first step outside spherical symmetry for AdS (in)stability problem has been done, and perturbation scheme for vacuum AdS has been constructed (stepping outside axial symmetry is more technical then conceptual issue).
- Each linearized eigenfrequency of AdS gives rise to aAdS, regular, time-periodic solutions of Einstein equations (geon), that are expected to be stable.
- One possible way to provide the evidence for the (dynamical) instability of AdS itself is to construct and analyze resonant approximation, the bases for this has been laid.

Final conclusions

- In fact the concepts gained from the study of nonlinear gravitational perturbations of AdS turned out to be robust and can be used in a broad context of gravitational perturbation problems (nonlinear gravitational waves, nonlinear quasinormal modes couplinings for Schwarzschild BH, and aftter including matter to the scheme also self-force, cosmological perturbations, etc.).
- **The hard part of perturbative Einstein equations (PDEs) can be reduced to only one scalar wave equation (for each polarization mode) and some linear algebra (!)**
- I am very grateful to CERN TH department for providing six quiet months (as scientific associate) to study gravitational perturbations of AdS.