Inverse Problems in the Era of Gravitational Wave Astronomy

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The Era of Gravitational Wave Astronomy

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- most convincing evidence for existence of black holes
- a new tool to study neutron stars and nuclear matter
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Among the many open problems ahead, is the exploration of the GW spectrum of compact relativistic objects.

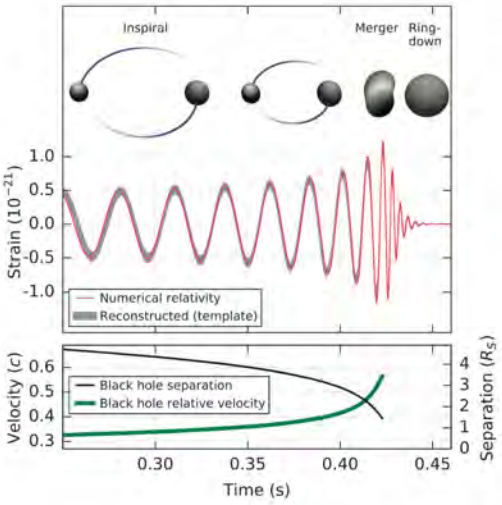

FIGURE 1: B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102, `https://doi.org/10.1103/PhysRevLett.116.061102`

PRIMER ON COMPACT OBJECTS

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Standard compact objects (within general relativity (GR)):

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Exotic compact objects (within or beyond GR):

- horizonless compact objects
- wormholes
- firewalls
- ...

PRIMER ON COMPACT OBJECTS

Standard compact objects (within general relativity (GR)):

- Clean tests of GR and the no-hair theorem
- Constraining the nuclear equation of state and stellar structure

[1] For a recent review: Cardoso and Pani [1]

Primer on Compact Objects

Standard compact objects (within general relativity (GR)):

- Clean tests of GR and the no-hair theorem
- Constraining the nuclear equation of state and stellar structure

Exotic compact objects[1] (within or beyond GR):

- What are possible smoking gun effects?
- If firewalls are real, could they be observable with GWs?
- Could there be new forms of matter under extreme conditions?

[1] For a recent review: Cardoso and Pani [1]

Oscillations of (compact, relativistic) objects have a long history:

- Black Holes
  - Schwarzschild BH: Regge and Wheeler 1957 [2], Zerilli 1970 [3]
  - Kerr BH: Teukolsky 1973 [4]
  - ...
- Neutron Stars
  - Thorne and Campolattaro 1967 [5]
  - ...

For classical reviews:

- Nollert 1999 [6]
- Kokkotas and Schmidt 1999 [7]
- Berti, Cardoso and Starinets 2009 [8]

EFFECTIVE WAVE EQUATION

For now, and to discuss the inverse problem later, we consider:

- spherical symmetry of $g_{\mu\nu}$
- no matter perturbations (or only axial modes in GR)

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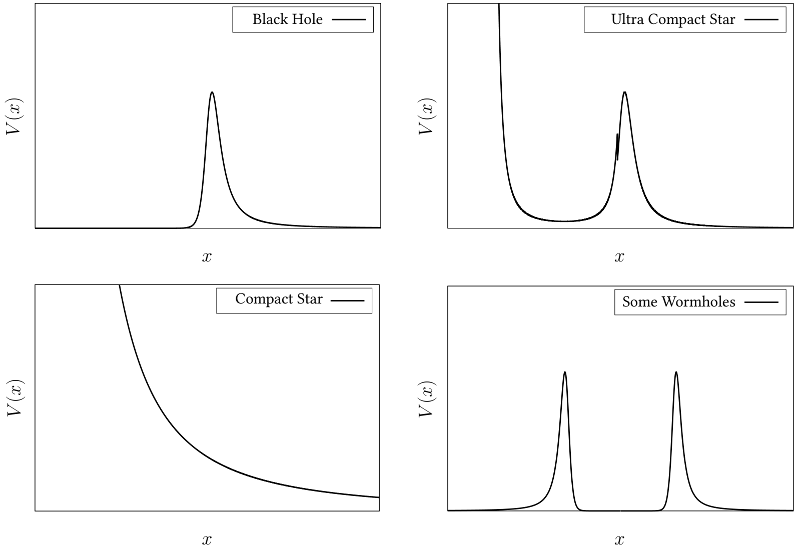
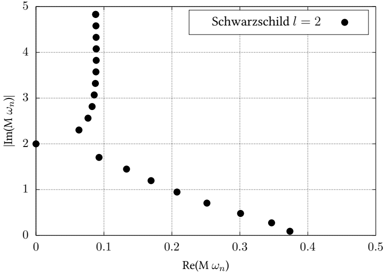
- **tortoise coordinate** $x \equiv \int \sqrt{g_{11}/g_{00}}\mathrm{d}r'$
- **effective potential** $V_l(r)$ (depends on $g_{\mu\nu}$)
- **quasi-normal modes** (QNMs) $\omega_n^2 \equiv E_n$ (complex valued)

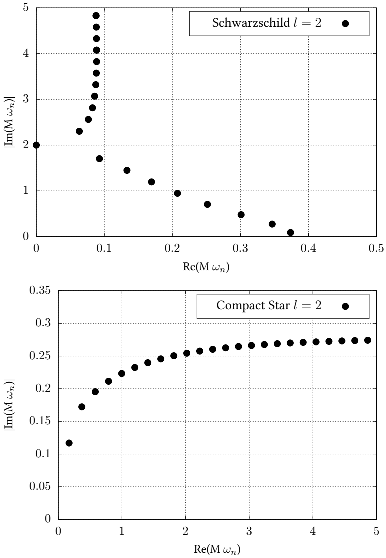
Perturbation Potentials of Compact Objects



QNM SPECTRA OF COMPACT OBJECTS



QNM spectra of Compact Objects



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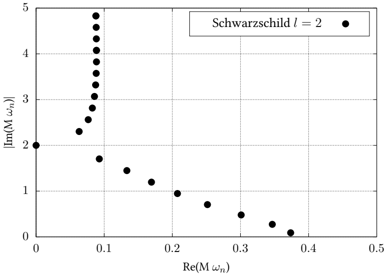

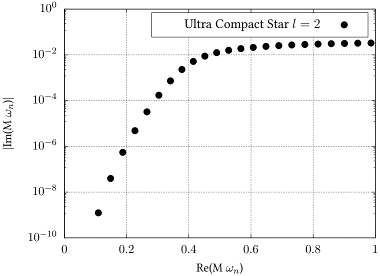

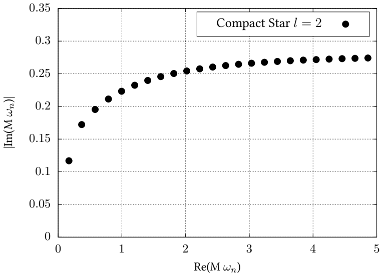

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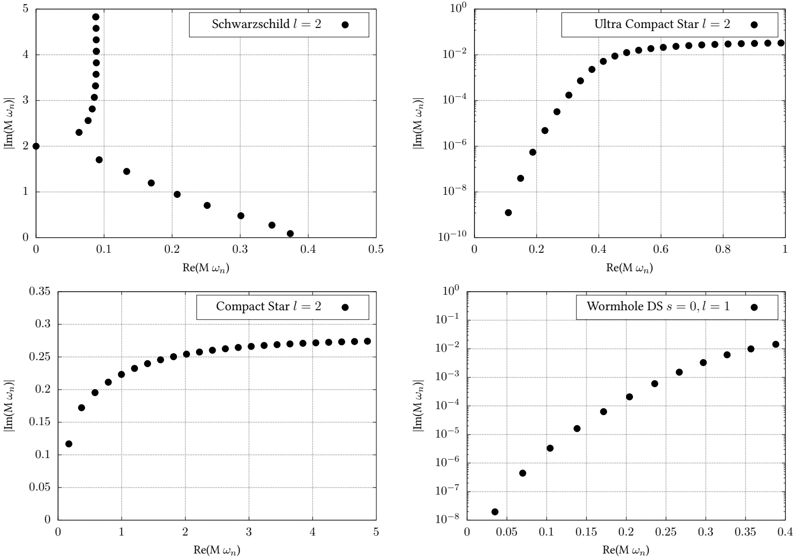
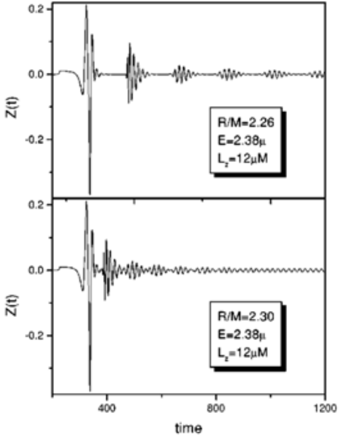
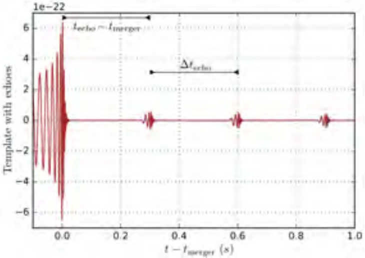


Trapped Modes alias Echoes





Left: Axial perturbations ultra compact stars, V. Ferrari and K. D. Kokkotas, Phys. Rev. D 62, 107504, 2000.
Right: Echoes from the abyss: Tentative evidence for Planck-scale structure at black hole horizons, Abedi, Dykaar and Afshordi, Phys. Rev. D 96, 082004 2017.

THE DIRECT PROBLEM

Determine QNM spectrum for a given potential:

$$V(r) \Rightarrow \omega_n \tag{2}$$

- Different types of approaches:
  - Approximate potentials[2]
  - WKB approximation[3]
  - phase integral method [4]
  - Leaver's continued fraction [21]
  - direct integration (root of Wronskian)
  - time evolution (numerical integration before $e^{i\omega t}$)

[2] Chandrasekhar and Detweiler [9], Ferrari and Mashhoon [10; 11], Blome and Masshoon [12]

[3] Schutz and Will [13], Iyer and Will [14], [15], Kokkotas and Schutz [16], Seidel and Iyer [17], Araujo, Nicholson and Schutz[18], Konoplya [19]

[4] Andersson, Araujo and Schutz[20]

The Inverse Problem

Use QNM spectrum to recover potential and source properties:

$$\omega_n \Rightarrow V(r) \tag{3}$$

THE INVERSE PROBLEM

Use QNM spectrum to recover potential and source properties:

$$\omega_n \Rightarrow V(r) \tag{3}$$

- Problem has changed significantly:
  - Well posed? Unique $V(r)$?

- Different types of approaches:
  - Brute force by repeatedly solving the direct problem.
  - Clever analysis of wave equation.

The Inverse Problem

Brute force by repeatedly solving the direct problem:

- propose some ansatz for the potential
- solve direct problem (using known methods)
- repeat until modified ansatz provides same spectrum

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- solve direct problem (using known methods)
- repeat until modified ansatz provides same spectrum

The good, the bad, and the ugly:

- only uses known methods
- uniqueness unclear
- ansatz unclear

Useful for numerical studies, optimization, parameter estimation

THE INVERSE PROBLEM

Clever analysis of wave equation:

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Clever analysis of wave equation:

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- the one we use is based on WKB

The good, the bad, and the ugly (using WKB):

- analytic study of the problem (approximate)
- WKB is only starting point, inversion relations have to be derived
- relies on validity of WKB, depends on type of potential

Matches analytic understanding with direct way of computation.

THE INVERSE PROBLEM: ASTEROSEISMOLOGY

Reconstruct neutron star properties from different types of oscillation modes:

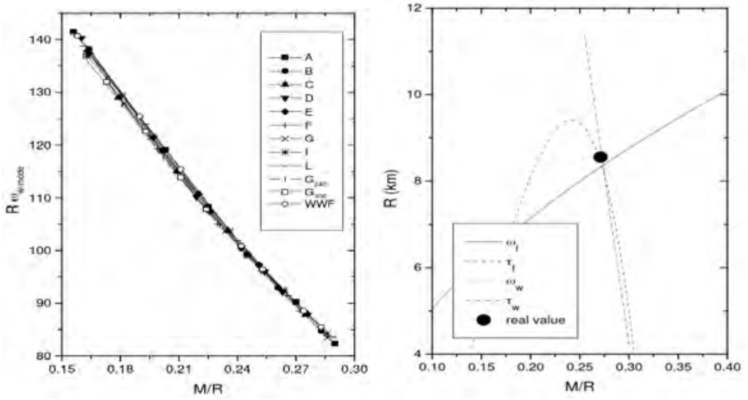


FIGURE 2: Taken from Andersson and Kokkotas, MNRAS, 299, 4, 1998, `https://doi.org/10.1046/j.1365-8711.1998.01840.x`

THE WKB METHOD

An asymptotic series for the wave function[5]

$$\Psi(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right), \tag{4}$$

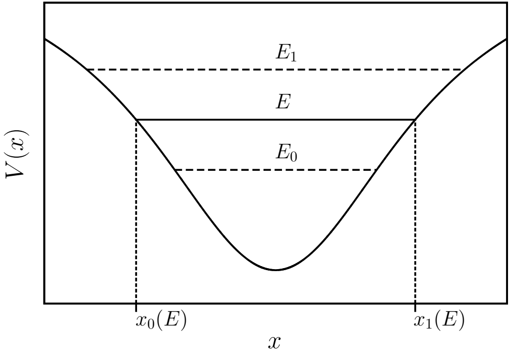
$$\begin{aligned}\Psi(x) = & c_1 Q^{-1/4}(x)\exp\left(\frac{1}{\varepsilon}\int_a^x \sqrt{Q(x')}\mathrm{d}x'\right) \qquad (5)\\ & + c_2 Q^{-1/4}(x)\exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{Q(x')}\mathrm{d}x'\right), \ \varepsilon \to 0, \qquad (6)\end{aligned}$$

[5] Standard reference: Bender and Orszag [22]

THE CLASSICAL BOHR SOMMERFELD RULE

$$\int_{x_0}^{x_1} \sqrt{E_n - V(x)} dx = \pi \left(n + \frac{1}{2} \right), \tag{7}$$

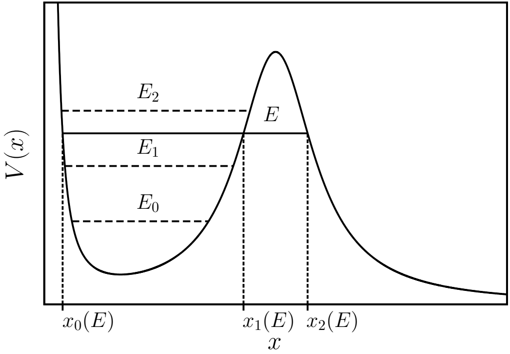
where (x_0, x_1) are the classical turning points and $n \in \mathbb{N}_0$.



GENERALIZED BOHR-SOMMERFELD RULES

Can be extended to other potentials [6]

$$\int_{x_0}^{x_1} \sqrt{E_n - V(x)}\mathrm{d}x = \pi\left(n + \frac{1}{2}\right) - \frac{i}{4}\exp\left(2i\int_{x_1}^{x_2}\sqrt{E_n - V(x)}\mathrm{d}x\right) \quad (8)$$



[6] Popov, Mur and Sergeev, Physics Letters A, 157, 4-5, 1991[23]

INVERTING BOHR-SOMMERFELD RULES

- Known for 2 turning point wells [7] or barriers[8]
- Extended to **quasi-stationary states** (3 or 4 turning points) [9]
- **Neutron star potentials** with discontinuity (1 turning point) [10]

$$\mathcal{L}_1(E) = x_1 - x_0 = 2\frac{\partial}{\partial E}\int_{E_{\min}}^{E} \frac{n(E') + 1/2}{\sqrt{E - E'}}\mathbf{d}E' \tag{9}$$

$$\mathcal{L}_2(E) = x_2 - x_1 = -\frac{1}{\pi}\int_{E}^{E_{\max}} \frac{(\mathbf{d}T(E')/\mathbf{d}E')}{T(E')\sqrt{E' - E}}\mathbf{d}E' \tag{10}$$

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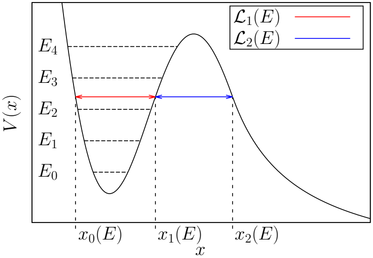
[7] Wheeler, 1976, [24]

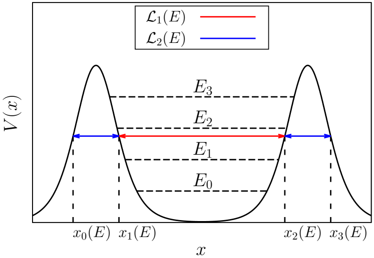
[8] WKB Gamow formula: Cole and Good, Phys. Rev. A, 18, 3, 1978, [25]

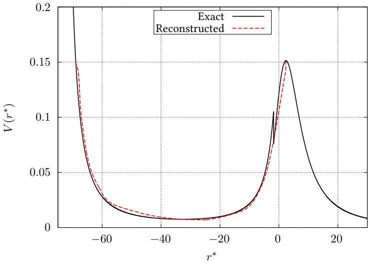
[9] Völkel and Kokkotas [26]; Völkel [27]; Völkel and Kokkotas [28]

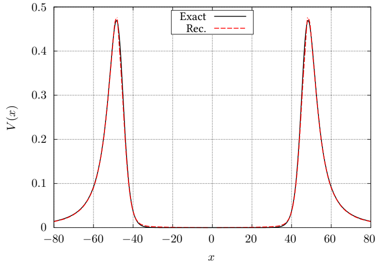
[10] Völkel and Kokkotas [29]

INVERTING BOHR-SOMMERFELD RULES









Inverse Problem for Neutron Stars

Using the axial QNM spectrum of non-rotating neutron stars[11]:

[11] *S. H. V. and K. D. Kokkotas, Class. Quantum Grav. 36, 115002, 2019, [29]*

Inverse Problem for Neutron Stars

Using the axial QNM spectrum of non-rotating neutron stars[11]:

- used WKB for this type of direct and inverse QNM problem
- applied to constant density and polytropic EOS
- approximate generalized rule (for real part of ω_n^2)

$$\int_{x_{0,1}}^{x_s} \sqrt{E_{0n} - V_1(x)}\mathrm{d}x = \pi\left(n + \frac{3}{4}\right) \tag{11}$$

[11] *S. H. V. and K. D. Kokkotas, Class. Quantum Grav. 36, 115002, 2019, [29]*

INVERSE PROBLEM FOR NEUTRON STARS

Two approximate educated guesses for the spectrum:

- suggesting fundamental mode ω_f related to potential at R

$$R\omega_f \approx \sqrt{\left(1-\frac{2M}{R}\right)\left(L-\frac{6M}{R}\right)} \approx \sqrt{L}-\left(\frac{3+L}{\sqrt{L}}\right)\frac{M}{R}+\mathcal{O}\left(\left(\frac{M}{R}\right)^2\right), \tag{12}$$

- approximation in the limit of large overtones[12], known empirically

$$\Delta\omega = \frac{\pi}{\mathcal{R}} \tag{13}$$

[12] confirms Zhang and Leung, Phys. Rev. D 83, 2011, [30]

INVERSE PROBLEM FOR NEUTRON STARS

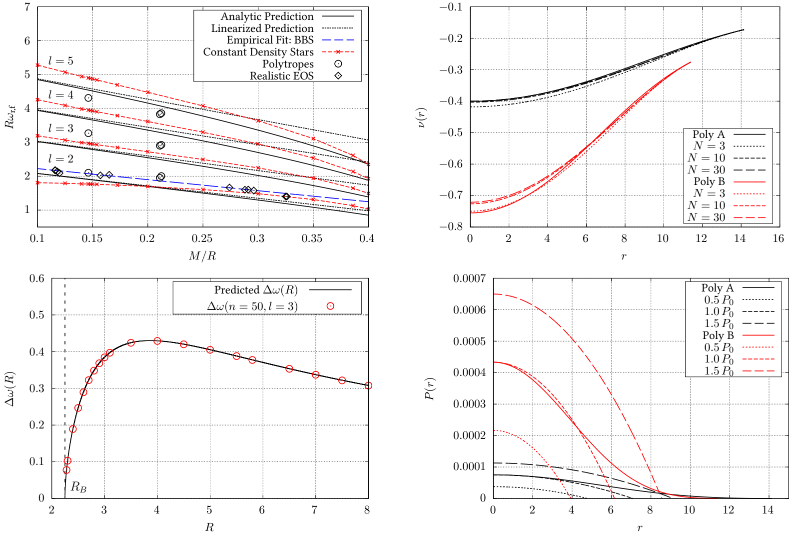


FIGURE 3: S. H. V. and K. D. Kokkotas, Class. Quantum Grav. 36, 115002, 2019, [29]

Uniqueness of Black Hole QNMs

How general and unique are Schwarzschild QNMs?

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bottom-up:

- use **parametrized ansatz for black hole metric**
- **no gravitational field equations**, only test fields
- compare Schwarzschild QNMs with parametrized metric QNMs
- **qualitative "general" conclusions**, not directly applicable to observations

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- compare Schwarzschild QNMs with parametrized metric QNMs
- **qualitative "general" conclusions**, not directly applicable to observations

top-down:

- choose **alternative theory of gravity**
- find **exact black hole solutions**, study **metric perturbations**
- compare Schwarzschild QNMs with alternative black hole QNMs
- **specific quantitative conclusions**, directly applicable to observations

Bottom-Up: Rezzolla-Zhidenko Metric

Parametrization for spherically symmetric black hole metric[13]

$$\mathrm{d}s^2 = -N^2(r)\mathrm{d}t^2 + \frac{B^2(r)}{N^2(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{14}$$

$$x \equiv 1 - \frac{r_0}{r}, \qquad N^2 = xA(x), \tag{15}$$

$$A(x) = 1 - \varepsilon(1-x) + (a_0 - \varepsilon)(1-x)^2 + \tilde{A}(x)(1-x)^3, \tag{16}$$

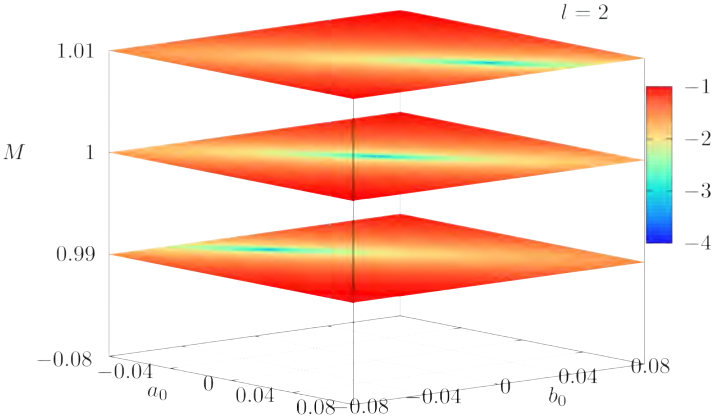
$$B(x) = 1 + b_0(1-x) + \tilde{B}(x)(1-x)^2. \tag{17}$$

Effective potential for scalar test field ($\Box\phi = 0$):

$$V_l(r) = \frac{l(l+1)}{r^2}N^2(r) + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r^*}\frac{N^2(r)}{B(r)}. \tag{18}$$

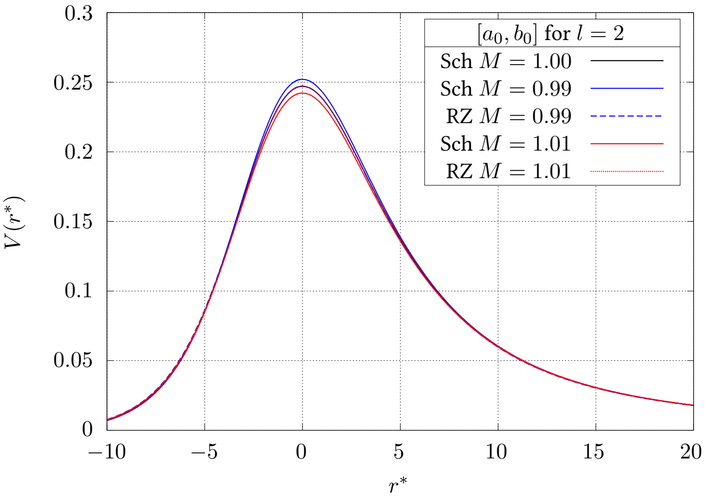
[13] Rezzolla and Zhidenko, Phys. Rev. D 90, 084009, 2014, [31]

Uniqueness of Black Hole QNMs



Relative error of combined real and imaginary part of ω_0 with respect to Schwarzschild ($M = 1$).
S. H. V. and K. D. Kokkotas, Phys. Rev. D, 100, 044026, 2019, [32]

UNIQUENESS OF BLACK HOLE QNMS



Effective potential for scalar field in the RZ background space-time. S. H. V. and K. D. Kokkotas, Phys. Rev. D, 100, 044026, 2019, [32]

UNIQUENESS OF BLACK HOLE QNMS

Ongoing work:

- include multiple modes simultaneously and account for uncertainties
- allow for multiple metric parameters (simultaneously)
- method of choice: Bayesian analysis using MCMC[14]
- yields posterior distribution for model parameters

[14] Markov-Chain-Monte-Carlo (MCMC)

UNIQUENESS OF BLACK HOLE QNMS

Basic preliminary results:

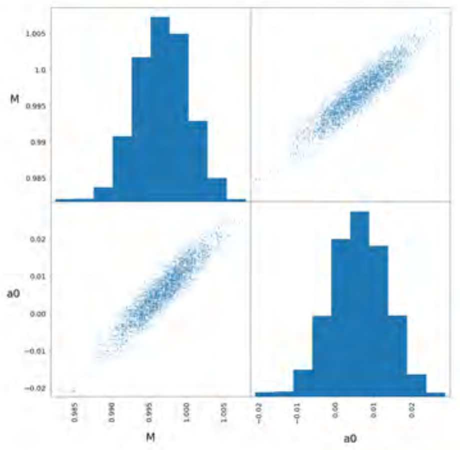


FIGURE 4: Model with $M = 1, a_0 = 0.01$ and Gaussian noise using (n=0,1; l=2) modes.

CONCLUSIONS

- Upcoming GW detections will provide several QNMs.
- Testing the no-hair theorem and neutron star physics is in reach.
- Calls for studying more general inverse spectrum problems.
- Semi-classical methods (like WKB) can provide some understanding.
- They are helpful for computationally expensive problems (MCMC).

BIBLIOGRAPHY I

[1] V. Cardoso and P. Pani. Testing the nature of dark compact objects: a status report. *Living Rev. Rel.*, 22(1):4, 2019. doi: 10.1007/s41114-019-0020-4.

[2] T. Regge and J. A. Wheeler. Stability of a Schwarzschild Singularity. *Physical Review*, 108: 1063–1069, November 1957. doi: 10.1103/PhysRev.108.1063.

[3] F. J. Zerilli. Effective Potential for Even-Parity Regge-Wheeler Gravitational Perturbation Equations. *Phys. Rev. Lett.*, 24:737–738, March 1970. doi: 10.1103/PhysRevLett.24.737.

[4] S. A. Teukolsky. Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations. *Astrophys. J.*, 185:635–647, 1973. doi: 10.1086/152444.

[5] K. S. Thorne and A. Campolattaro. Non-Radial Pulsation of General-Relativistic Stellar Models. I. Analytic Analysis for L $\geq$ 2. *ApJ*, 149:591, September 1967. doi: 10.1086/149288.

[6] H.-P. Nollert. Quasinormal modes: the characteristic sound of black holes and neutron stars. *Class. Quantum Grav.*, 16(12):R159–R216, nov 1999. doi: 10.1088/0264-9381/16/12/201.

[7] K. D. Kokkotas and B. G. Schmidt. Quasi-Normal Modes of Stars and Black Holes. *Living Rev. Rel.*, 2:2, December 1999. doi: 10.12942/lrr-1999-2.

[8] E. Berti, V. Cardoso, and A. O. Starinets. Quasinormal modes of black holes and black branes. *Class. Quantum Grav.*, 26:163001, 2009. doi: 10.1088/0264-9381/26/16/163001.

[9] S. Chandrasekhar and S. Detweiler. The quasi-normal modes of the Schwarzschild black hole. *Proceedings of the Royal Society of London Series A*, 344:441–452, August 1975. doi: 10.1098/rspa.1975.0112.

BIBLIOGRAPHY II

[10] V. Ferrari and B. Mashhoon. Oscillations of a black hole. *Phys. Rev. Lett.*, 52:1361–1364, April 1984. doi: 10.1103/PhysRevLett.52.1361.

[11] V. Ferrari and B. Mashhoon. New approach to the quasinormal modes of a black hole. *Phys. Rev. D*, 30:295–304, July 1984. doi: 10.1103/PhysRevD.30.295.

[12] H.-J. Blome and B. Mashhoon. Quasi-normal oscillations of a schwarzschild black hole. *Phys. Lett. A*, 100:231–234, January 1984. doi: 10.1016/0375-9601(84)90769-2.

[13] B. F. Schutz and C. M. Will. Black hole normal modes - A semianalytic approach. *ApJ*, 291: L33–L36, April 1985. doi: 10.1086/184453.

[14] S. Iyer and C. M. Will. Black-hole normal modes: A WKB approach. I. Foundations and application of a higher-order WKB analysis of potential-barrier scattering. *Phys. Rev. D*, 35: 3621–3631, June 1987. doi: 10.1103/PhysRevD.35.3621.

[15] S. Iyer. Black-hole normal modes: A wkb approach. ii. schwarzschild black holes. *Phys. Rev. D*, 35:3632–3636, Jun 1987. doi: 10.1103/PhysRevD.35.3632.

[16] K. D. Kokkotas and B. F. Schutz. Black-hole normal modes: A wkb approach. iii. the reissner-nordström black hole. *Phys. Rev. D*, 37:3378–3387, Jun 1988. doi: 10.1103/PhysRevD.37.3378.

[17] E. Seidel and S. Iyer. Black-hole normal modes: A wkb approach. iv. kerr black holes. *Phys. Rev. D*, 41:374–382, Jan 1990. doi: 10.1103/PhysRevD.41.374.

BIBLIOGRAPHY III

[18] M E Araujo, D Nicholson, and B F Schutz. On the bohr-sommerfeld formula for black hole normal modes. *Classical and Quantum Gravity*, 10(6):1127–1138, jun 1993. doi: 10.1088/0264-9381/10/6/009. URL https://doi.org/10.1088%2F0264-9381%2F10%2F6%2F009.

[19] R. A. Konoplya. Quasinormal behavior of the *d*-dimensional schwarzschild black hole and the higher order wkb approach. *Phys. Rev. D*, 68:024018, Jul 2003. doi: 10.1103/PhysRevD.68.024018.

[20] N Andersson, M E Araujo, and B F Schutz. The phase-integral method and black hole normal modes. *Classical and Quantum Gravity*, 10(4):735–755, apr 1993. doi: 10.1088/0264-9381/10/4/009. URL https://doi.org/10.1088%2F0264-9381%2F10%2F4%2F009.

[21] E. W. Leaver. An analytic representation for the quasi-normal modes of Kerr black holes. *Proceedings of the Royal Society of London Series A*, 402:285–298, December 1985. doi: 10.1098/rspa.1985.0119.

[22] C. M. Bender and S. A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. New York: McGraw-Hill, 1978.

[23] V. S. Popov, V. D. Mur, and A. V. Sergeev. Quantization rules for quasistationary states. *Phys. Lett. A*, 157:185–191, July 1991. doi: 10.1016/0375-9601(91)90048-D.

[24] J. A. Wheeler. *Studies in Mathematical Physics: Essays in Honor of Valentine Bargmann*. Princeton Series in Physics. Princeton University Press, 2015. ISBN 9780608066288.

BIBLIOGRAPHY IV

[25] M. W. Cole and R. H. Good. Determination of the shape of a potential barrier from the tunneling transmission coefficient. *Phys. Rev. A*, 18:1085–1088, Sep 1978. doi: 10.1103/PhysRevA.18.1085.

[26] S. H. Völkel and K. D. Kokkotas. Ultra Compact Stars: Reconstructing the Perturbation Potential. *Class. Quantum Grav.*, 34(17):175015, 2017. doi: 10.1088/1361-6382/aa82de.

[27] S. H. Völkel. Inverse spectrum problem for quasi-stationary states. *J. Phys. Commun.*, 2(2): 025029, 2018. doi: 10.1088/2399-6528/aaaee2.

[28] S. H. Völkel and K. D. Kokkotas. Wormhole potentials and throats from quasi-normal modes. *Class. Quantum Grav.*, 35(10):105018, May 2018. doi: 10.1088/1361-6382/aabce6.

[29] S. H. Völkel and K. D. Kokkotas. On the inverse spectrum problem of neutron stars. *Class. Quantum Grav.*, 36(11):115002, Jun 2019. doi: 10.1088/1361-6382/ab186e.

[30] Y. J. Zhang, J. Wu, and P. T. Leung. High-frequency behavior of *w*-mode pulsations of compact stars. *Phys. Rev. D*, 83:064012, Mar 2011. doi: 10.1103/PhysRevD.83.064012.

[31] L. Rezzolla and A. Zhidenko. New parametrization for spherically symmetric black holes in metric theories of gravity. *Phys. Rev. D*, 90:084009, Oct 2014. doi: 10.1103/PhysRevD.90.084009.

[32] S. H. Völkel and K. D. Kokkotas. Scalar Fields and Parametrized Spherically Symmetric Black Holes: Can one hear the shape of space-time? *Phys. Rev.*, D100(4):044026, 2019. doi: 10.1103/PhysRevD.100.044026.